Physikalisches Institut
Universität Bonn
Theoretische Physik

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Exercise 7

Exercises General Relativity and Cosmology

Prof. Dr. Albrecht Klemm

1 The precession of Mercury — 8 points

The metric for the Schwarzschild solution reads

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2, \tag{1.1}$$

where $d\Omega^2$ denotes the standard metric on a two-sphere.

1. Show using the variational calculus (or equivalently the Euler-Lagrange equations) that the equations for the geodesics read

$$\begin{aligned}
\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} &= 0,\\
\frac{d^2r}{d\lambda^2} + \frac{GM}{r^3}(r-2GM)\Big(\frac{dt}{d\lambda}\Big)^2 - \frac{GM}{r(r-2GM)}\Big(\frac{dr}{d\lambda}\Big)^2 &\\
-(r-2GM)\left(\Big(\frac{d\theta}{d\lambda}\Big)^2 + \sin^2\theta\Big(\frac{d\phi}{d\lambda}\Big)^2\right) &= 0,\\
\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\Big(\frac{d\phi}{d\lambda}\Big)^2 &= 0,\\
\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} &= 0.
\end{aligned} \tag{1.2}$$

2. Read off all Christoffel symbols.

3. Show explicitly that

$$K_\mu = \Big(-\Big(1-\frac{2GM}{r}\Big), 0, 0, 0\Big), \qquad R_\mu = \Big(0, 0, 0, r^2\sin^2\theta\Big) \tag{1.3}$$

are Killing vector fields.

4. Show that there are two more Killing vectors that allow to restrict ourselves to geodesics of the form

$$\theta = \frac{\pi}{2}. \tag{1.4}$$

5. Show that

$$E = \Big(1-\frac{2GM}{r}\Big)\frac{dt}{d\lambda}, \qquad L = r^2\frac{d\phi}{d\lambda} \tag{1.5}$$

are conserved quantities. Which physical quantities do they correspond to?

6. Using that
$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{1.6}$$
is constant along a path show that
$$-\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2+\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2+r^2\left(\frac{d\phi}{d\lambda}\right)^2=-\epsilon \tag{1.7}$$
can be rewritten as
$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2+V(r)=\mathcal{E}, \tag{1.8}$$
where we have defined
$$V(r)=\frac{1}{2}\epsilon-\epsilon\frac{GM}{r}+\frac{L^2}{2r^2}-\frac{GML^2}{r^3}, \quad \mathcal{E}=\frac{1}{2}E^2. \tag{1.9}$$

7. Show that (1.9) may be rewritten as
$$\left(\frac{dr}{d\phi}\right)^2+\frac{1}{L^2}r^4-\frac{2GM}{L^2}r^3+r^2-2GMr=\frac{2\mathcal{E}}{L^2}r^4. \tag{1.10}$$
Introducing
$$x=\frac{L^2}{GMr}, \tag{1.11}$$
show that (1.10) can be brought into the form
$$\frac{d^2x}{d\phi^2}-1+x=\frac{3G^2M^2}{L^2}x^2. \tag{1.12}$$
If the last term was absent we would obtain the result of a Newtonian calculation. As a next step we split x into a Newtonian solution plus a perturbation
$$x=x_0+x_1 \tag{1.13}$$

8. Show that the zeroth order part respectively the first order part read
$$\frac{d^2x_0}{d\phi^2}-1+x_0=0, \qquad \frac{d^2x_1}{d\phi^2}+x_1=\frac{3G^2M^2}{L^2}x_0^2. \tag{1.14}$$

9. Show that the zeroth order solution is the Kepler solution
$$x_0=1+e\cos\phi. \tag{1.15}$$
In addition show that a solution for the perturbation part is given by
$$x_1=\frac{3G^2M^2}{L^2}\left(\left(1+\frac{1}{2}e^2\right)+e\phi\sin\phi-\frac{1}{6}e^2\cos 2\phi\right). \tag{1.16}$$

In this solution we are for the moment only interested in the second term. Show that in this particular case the full solution can be expressed as

$$x = 1 + e\cos\big((1-\alpha)\phi\big), \qquad \alpha = \frac{3G^2M^2}{L^2}. \tag{1.17}$$

Show that during each orbit of the planet the perihelion advances by an angle

$$\Delta\phi = 2\pi\alpha = \frac{6\pi G^2M^2}{L^2}. \tag{1.18}$$

Here we approximate

$$L^2 \approx GM(1-e^2)a. \tag{1.19}$$

The relevant data for the orbit of Mercury read

$$\frac{GM}{c^2} = 1.48\times 10^{15}\mathrm{cm}, \quad a = 5.79\times 10^{12}\mathrm{cm}, \quad e = 0.2056. \tag{1.20}$$

10. Calculate the total precession of Mercury in one century.

2 Conservation of the energy-momentum tensor **2 points**

The variation of the metric under a diffeomorphism generated by the vector feld V^μ is given by

$$\delta g_{\mu\nu} = \mathcal{L}_V g_{\mu\nu}. \tag{2.1}$$

Show that demanding the vanishing of the variation of the sum of Einstein Hilbert action and the matter action, which is given as

$$S = \frac{1}{16\pi G}S_H[g_{\mu\nu}] + S_M[g_{\mu\nu},\psi^i], \tag{2.2}$$

implies energy-momentum conservation

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.3}$$