Exercises General Relativity and Cosmology

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1 Conformal transformations **3 points**

Let M be a manifold equipped with a metric g. A conformal transformation multiplies the metric by a space-time dependent, non-vanishing function ω

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}. \tag{1.1}$$

1. Show that a conformal transformation maps null-curves to null-curves.
2. Show that a conformal transformation preserves the angle.
3. The connections of $\tilde{g}$ and g are related by

$$\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + C^\rho_{\mu\nu} \tag{1.2}$$

   where $C^\rho_{\mu\nu}$ transforms as a tensor. Show that

$$C^\rho_{\mu\nu} = \omega^{-1}\left(\delta^\rho_\mu \nabla_\nu \omega + \delta^\rho_\nu \nabla_\mu \omega - g_{\mu\nu} g^{\rho\lambda} \nabla_\lambda \omega\right) \tag{1.3}$$

4. Show that a conformal transformation maps null-geodesics to null-geodesics. Hint: How are the affine parameters related?

2 Conformal diagrams **3 points**

The goal of this exercise is to explore how to use coordinates such that one can display an initially infinite space-time by a finite coordinate range. This can be done by performing suitable conformal transformations. The price one has to pay is that the metric becomes un-physical, however as you have shown in the previous exercise the causal structure remains unaffected.
In the following we consider Minkowski space-time using polar coordinates, i.e. our metric reads

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{2.1}$$

where $d\Omega^2$ denotes the canonical metric on a two-sphere. We introduce null coordinates

$$u = t - r, \quad v = t + r, \qquad -\infty < u < \infty, \quad -\infty < v < \infty, \quad u \leq v. \tag{2.2}$$

Show that the Minkowski metric reads in these coordinates

$$ds^2 = -du dv + \frac{1}{4}(v-u)^2 d\Omega^2. \tag{2.3}$$

We introduce yet another set of coordinates given by

$$U = \arctan u, \quad V = \arctan v, \qquad -\frac{\pi}{2} < U < \frac{\pi}{2}, \quad -\frac{\pi}{2} < V < \frac{\pi}{2}, \quad U \leq V. \tag{2.4}$$

Show that the metric reads in these coordinates

$$ds^2 = \frac{1}{4\cos^2 U \cos^2 V}\Big(-4dUdV + \sin^2(V-U)d\Omega^2\Big). \tag{2.5}$$

Re-introducing time- respectively space-like coordinates

$$T = U + V, \quad R = V - U, \qquad 0 \leq R < \pi, \quad |T| + R < \pi, \tag{2.6}$$

the metric reads

$$ds^2 = \frac{1}{\big(\cos T + \cos R\big)^2}\Big(-dT^2 + dR^2 + \sin^2 R d\Omega^2\Big). \tag{2.7}$$

Show that this is conformally equivalent to the manifold $S^3 \times \mathbb{R}^1$, provided we allow any real value for T. Show that the Minkowski space is given as a triangle in the coordinates T, R. Mark

- future time-like infinity ($t = \infty$, $r = 0$)
- spatial infinity ($t = 0$, $r = \infty$)
- past time-like infinity ($t = -\infty$, $r = 0$)
- future null infinity ($t = \infty$, $0 < r < \infty$)
- past null infinity ($t = -\infty$, $0 < r < \infty$)

Furthermore evaluate the lines of constant r and t and show that light cones are at 45°.
Hint: It is not sufficient to simply copy figure H.4 from Carroll, but you are asked to derive it!

3 Kruskal coordinates **4 points**

We consider the Schwarzschild metric

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1}dr^2 + r^2 d\Omega^2. \tag{3.1}$$

In these coordinates the metric gets singular at the event horizon $r = 2GM$. Actually one can show that this is not a physical singularity but merely a choice of bad coordinates. In this exercise we want to find "good" coordinates to explore the region behind the event horizon.

1. Introducing the coordinate

$$r^* = r + 2GM \ln\Big(\frac{r}{2GM} - 1\Big), \tag{3.2}$$

show that the Schwarzschild metric becomes in these coordinates

$$ds^2 = \Big(1 - \frac{2GM}{r}\Big)\big(-dt^2 + dr^{*2}\big) + r^2(r^*)d\Omega^2. \tag{3.3}$$

2. We introduce coordinates v, u corresponding to in-going respectively out-going null-geodesics by
$$v = t + r^*, \quad u = t - r^*. \tag{3.4}$$
As a next step we re-define
$$v' = e^{v/4GM}, \quad u' = -e^{-u/4GM}. \tag{3.5}$$
Show that the metric reads in these coordinates
$$ds^2 = -\frac{32G^3M^3}{r} e^{-r/2GM} dv'du' + r^2 d\Omega^2. \tag{3.6}$$

3. Finally, the so-called Kruskal coordinates read
$$T = \frac{1}{2}(v' + u'), \quad R = \frac{1}{2}(v' - u'). \tag{3.7}$$
Show that the metric is given as
$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM} \big(- dT^2 + dR^2 \big) + r^2 d\Omega^2. \tag{3.8}$$
Note that r is a function of T, R. Is there any problem for $r = 2GM$?

4. Now draw the so-called Kruskal diagram, i.e. draw a coordinate system with respect to R, T (i.e. (almost) every point corresponds to an S^2) .
   - How do radial null-curves look like?
   - Draw lines of constant r.
   - Draw lines of constant t.
   - Draw the event horizon for $t = \infty$, as well as for $t = -\infty$.
   - Mark the regions where R, T are no good coordinates.
   - Evaluate the light-cone for a given point in the coordinate system.