Tutorial Conformal Field Theory — Sheet 1

1.) **Anharmonic ratios** `4 pt`

We consider conformal invariant ratios in $\mathbb{R}^d$ with $d > 2$.

a) Given four distinct points $x_1, \ldots, x_4$ show that the two anharmonic ratios,

$$\frac{|x_1 - x_2| \cdot |x_3 - x_4|}{|x_1 - x_3| \cdot |x_2 - x_4|} \,, \qquad \frac{|x_1 - x_2| \cdot |x_3 - x_4|}{|x_1 - x_4| \cdot |x_2 - x_3|} \,,$$

are conformal invariants.

b) Given N distinct points show that there are $\frac{1}{2}N(N-3)$ independent anharmonic ratios.

2.) **Conformal transformations** `6 pt`

Let us examine the group of conformal transformations acting on the Euclidean space $\mathbb{R}^d$ for $d > 2$. The Lie group $SO(1, d+1)$ acts naturally on $\mathbb{R}^{d+2}$. Consider the map

$$\iota : \mathbb{R}^d \to \mathbb{RP}^{d+1} \,, \; x^\mu \to \left[\tfrac{1}{2}(1 + x^2) \, : \, x^1 \, : \, \ldots \, : \, x^d \, : \, \tfrac{1}{2}(1 - x^2) \right] \,,$$

which induces a $SO(1, d+1)$ action on $\mathbb{R}^d$.

a) Show that the (connected component with the identity of the) Lie group $SO(1, d+1)$ has the same number of generators as the conformal group of $\mathbb{R}^d$.

b) Check that ι maps points in $\mathbb{R}^d$ onto the projective light cone of $\mathbb{PR}^{d+1}$.

c) Demonstrate that the $SO(1, d+1)$ matrices

$$\begin{pmatrix} 1 & & \\ & \mathbf{\Lambda}_{d\times d} & \\ & & 1 \end{pmatrix} , \qquad \begin{pmatrix} \frac{1+r^2}{2r} & 0 & \frac{1-r^2}{2r} \\ 0 & \mathbf{1}_{d\times d} & 0 \\ \frac{1-r^2}{2r} & 0 & \frac{1+r^2}{2r} \end{pmatrix} ,$$

$$\begin{pmatrix} 1 + \frac{a^2}{2} & -\vec{a} & \frac{a^2}{2} \\ -\vec{a} & \mathbf{1}_{d\times d} & -\vec{a} \\ -\frac{a^2}{2} & \vec{a} & 1 - \frac{a^2}{2} \end{pmatrix} , \qquad \begin{pmatrix} 1 + \frac{b^2}{2} & -\vec{b} & -\frac{b^2}{2} \\ -\vec{b} & \mathbf{1}_{d\times d} & \vec{b} \\ \frac{b^2}{2} & -\vec{b} & 1 - \frac{b^2}{2} \end{pmatrix} ,$$

with $\mathbf{\Lambda}_{d\times d} \in SO(d)$, $r > 0$ and $\vec{a}, \vec{b} \in \mathbb{R}^d$, map to rotations, dilatations, translations and special conformal transformations on $\mathbb{R}^d$, respectively.