Tutorial Conformal Field Theory — Sheet 2

1.) **Correlation functions of quasi-primary fields** 8 pt

Let us consider correlation function of quasi-primary fields $\phi_k(\vec{x})$ with scaling dimension Δ_k in d dimensions ($d > 2$). We consider conformal invariant ratios in $\mathbb{R}^d$ with $d > 2$.

a) Show that the three-point correlations function is given by

$$\langle \phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\phi_3(\vec{x}_3)\rangle \;=\; \frac{C_{123}}{|\vec{x}_1-\vec{x}_2|^{\Delta_1+\Delta_2-\Delta_3}|\vec{x}_1-\vec{x}_3|^{\Delta_1+\Delta_3-\Delta_2}|\vec{x}_2-\vec{x}_3|^{\Delta_2+\Delta_3-\Delta_1}} \;,$$

with an undetermined real constant C_{123}.

b) Show that the four-point correlation function takes the form

$$\langle \phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\phi_3(\vec{x}_3)\phi_4(\vec{x}_4)\rangle \;=\; f(A_1,A_2)\prod_{1\le k<l\le 4}|\vec{x}_k-\vec{x}_l|^{\Delta/3-\Delta_k-\Delta_l} \;,$$

where $\Delta = \Delta_1 + \ldots + \Delta_4$ and f is an undetermined function of the two anharmonic ratios

$$A_1 \;=\; \frac{|\vec{x}_1-\vec{x}_2|\cdot|\vec{x}_3-\vec{x}_4|}{|\vec{x}_1-\vec{x}_3|\cdot|\vec{x}_2-\vec{x}_4|} \;, \qquad A_2 \;=\; \frac{|\vec{x}_1-\vec{x}_2|\cdot|\vec{x}_3-\vec{x}_4|}{|\vec{x}_1-\vec{x}_4|\cdot|\vec{x}_2-\vec{x}_3|} \;.$$

2.) **Scale invariance in momentum space** 2 pt

We consider a theory that is scale invariant in d dimensions, i.e., the theory is invariant under translations, rotations, and dilatations. In momentum space, a correlation function of N scalar field $\phi_i(\vec{x}_i)$, $i = 1, \ldots, N$, is represented by the Fourier transform $C(\vec{k}_1, \ldots, \vec{k}_n)$

$$\langle \phi_1(\vec{x}_1)\cdots\phi_N(\vec{x}_N)\rangle \;=\; \int \frac{d\vec{k}_1}{(2\pi)^d}\cdot\ldots\cdot\frac{d\vec{k}_{N-1}}{(2\pi)^d}e^{i(\vec{k}_1\cdot\vec{x}_1+\ldots+\vec{k}_N\cdot\vec{x}_N)}C(\vec{k}_1,\ldots,\vec{k}_n) \;,$$

where $-\vec{k}_N = \vec{k}_1 + \ldots + \vec{k}_{N-1}$ is fixed by momentum conservation (translation invariance).

a) Show that scale invariance of the theory implies

$$C(\vec{k}_1,\ldots,\vec{k}_n) \;=\; \Lambda^{(N-1)d-\Delta_1-\ldots-\Delta_N}C(\Lambda\vec{k}_1,\ldots,\Lambda\vec{k}_n) \;,$$

where Δ_i are the scaling dimensions of the fields ϕ_i.

b) Given the two point correlations $\langle\phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\rangle \sim |\vec{x}_1-\vec{x}_2|^{-(d-2+\eta)}$ (with the critical exponent η), show that the two-point correlator in momentum space is given by

$$C(\vec{k},-\vec{k}) \sim \frac{1}{|\vec{k}|^{2-\eta}} \;.$$

c) `Bonus Problem` `2 bonus pt`

Consider now the special case of a scale invariant theory in two dimensions, i.e., $d = 2$. Introducing an infrared cutoff at length scale L, show that the two-point function in coordinate space is proportional to

$$G_L(r) = \int_{L^{-1}}^{+\infty}\frac{dk}{k^{1-\eta}}J_0(kr) \;,$$

where $r = |\vec{x}_1 - \vec{x}_2|$ and J_0 is the zeroth-order Bessel function. Explain how this expression is compatible with two-point correlation function discussed in the lecture.