Tutorial Conformal Field Theory **Sheet 3**

1.) **Properties of global conformal transformations in two dimensions** 6 pt
We want to show that the group $SL(2,\mathbb{C})/\mathbb{Z}_2$ of global conformal transformations in two dimensions — also called the projective special linear group $PSL(2,\mathbb{C})$ — is isomorphic to the restricted four-dimensional Lorentz group $SO_+(1,3)$. (Recall that the restricted Lorentz group $SO_+(1,3)$ is the group of linear transformations on a four-vector x^μ that leaves the square of the norm $|x^\mu|^2 \equiv -(x^0)^2 + (x^1)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$ invariant and that preserves both orientation of space and direction of time.)

a) To any four-vector x^μ we associate a 2×2-matrix $X = \sum_{\mu=0}^3 x^\mu \sigma_\mu$ with σ_0 the identity and $\sigma_1, \sigma_2, \sigma_3$ the Pauli matrices. Show that $|x^\mu|^2 = -\det X$ and that any transformation $X \mapsto S^\dagger X S$ leaves the square of the norm invariant if S is a $SL(2,\mathbb{C})$ transformation.

b) Argue that any restricted Lorentz transformation can be associated to a $SL(2,\mathbb{C})$ matrix.

c) Analyze which $SL(2,\mathbb{C})$ transformations leave the matrix X invariants. Conclude that the group $SL(2,\mathbb{C})/\mathbb{Z}_2$ is isomorphic to the restricted Lorentz group $SO_+(1,3)$. Argue that therefore the group of global conformal transformations in two dimensions — also known as Möbius transformations —

$$z \mapsto \frac{az+b}{cz+d} \ , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \ ,$$

are isomorphic to the restricted Lorentz group $SO_+(1,3)$.

d) Determine the Möbius transformation that maps three arbitrary but distinct points $p_1, p_2, p_3 \in \mathbb{C}$ to the points 0, 1 and ∞ on the compactified complex plane $\mathbb{C} \cup \infty$.

e) For infinitesimal Möbius transformations, i.e.,

$$z \mapsto \frac{(1+\delta_1)z + \delta_2}{\delta_3 z + (1+\delta_4)} \ , \quad \delta_1, \delta_2, \delta_3, \delta_4 \text{ infinitesimal} \ ,$$

determine the variation $\epsilon(z)$ of the infinitesimal global conformal transformation $z \mapsto \tilde{z} = z + \epsilon(z)$.

2.) **Clustery property** 4 pt
Consider a generic four-point function of quasi-primary fields $\phi_k(z_k, \bar{z}_k)$, $k = 1,2,3,4$, in a two-dimensional conformal field theory:

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4)\rangle = f(\eta,\bar{\eta}) \prod_{1\leq k<l\leq 4} (z_k - z_l)^{h/3 - h_k - h_l} (\bar{z}_k - \bar{z}_l)^{\bar{h}/3 - \bar{h}_k - \bar{h}_l}$$

with $h_k > 0$ and $\bar{h}_k > 0$ for all k and

$$h = \sum_{k=1}^4 h_k \ , \quad \bar{h} = \sum_{k=1}^4 \bar{h}_k \ , \quad \eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

Show that a product of two-point functions is recovered in the limit when four points are paired in such a way that the two points in each pair are much closer to each other than the distance between the pairs.