Tutorial Conformal Field Theory — Sheet 5

1.) **Four-point correlation function of the free boson** 4 pt

Let us consider the two-dimensional free conformal field theory of a real boson.

a) Using Wick's theorem calculate the four-point correlation function

$$\langle \partial_{z_1}\phi(z_1,\bar z_1)\partial_{z_2}\phi(z_2,\bar z_2)\partial_{z_3}\phi(z_3,\bar z_3)\partial_{z_4}\phi(z_4,\bar z_4)\rangle$$

of the primaries of $\partial_z\phi(z,\bar z)$.

b) Compare the result with the general expression for four-point correlators

$$\langle \phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\phi_3(z_3,\bar z_3)\phi_4(z_4,\bar z_4)\rangle = f(\eta,\bar\eta)\prod_{1\le k<l\le 4}(z_k-z_l)^{h/3-h_k-h_l}(\bar z_k-\bar z_l)^{\bar h/3-\bar h_k-\bar h_l}$$

of the quasi-primaries $\phi_k(z,\bar z)$ of conformal weight h_k and $\bar h_k$ with

$$h=\sum_{k=1}^4 h_k\ , \quad \bar h=\sum_{k=1}^4 \bar h_k\ , \quad \eta=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}\ .$$

Determine the function $f(\eta,\bar\eta)$.

2.) **Generalized ghost system** 3 pt

Consider the two-dimensional conformal field theory of a pair of anti-commuting ghost fields $b(z)$ and $c(z)$ with the OPE

$$c(z)b(w)\ \sim\ \frac{1}{z-w}\ ,$$

and the holomorphic energy momentum tensor

$$T(z)\ =\ (1-\lambda)\ :(\partial_z b(z))c(z):\ -\lambda\ :b(z)(\partial_z c(z)):\ .$$

a) Show that the ghost fields $b(z)$ and $c(z)$ have conformal weight λ and $1-\lambda$.

b) Show that the central charge of this system is $c=-2(6\lambda^2-6\lambda+1)$.

c) Which conformal field theory is identified with the generalized ghost system for $\lambda=\frac{1}{2}$.

d) For $\lambda=2$ we obtain the reparametrization ghost system of the bosonic string. Quantum-mechanically a consistent string theory requires a worldsheet conformal field theory of vanishing central charge. The worldsheet of the bosonic string in D-dimensional Minkowski space $\mathbb{R}^{1,D-1}$ is described by D free bosons together with the reparametrization ghost system. Determine the critical dimension D of Minkowski space, which yields a consistent bosonic string theory on the quantum level.

3.) **The Schwarzian derivative** 3 pt

Given to two successive local conformal transformations $z\to w(z)$ and $w\to u(w)$, demonstrate that the Schwarzian derivative fulfills the group property

$$\{u;z\}\ =\ \{w;z\}+\left(\frac{dw}{dz}\right)^2\{u;w\}\ .$$