1.) **Correlation functions of descendant fields**   2 pt
Give a general formula for the correlation function

$$\left\langle \phi_0^{\{\vec{k}\}\{\vec{\bar{k}}\}}(w,\bar{w})\, \phi_1(w_1,\bar{w}_1)\, \ldots\, \phi_N(w_N,\bar{w}_N) \right\rangle$$

of one arbitrary descendant field $\phi_0^{\{\vec{k}\}\{\vec{\bar{k}}\}}(w,\bar{w})$ with a product of primary fields $\phi_\ell(w_\ell,\bar{w}_\ell)$, $\ell = 1,\ldots,N$, in terms of the primary correlator

$$\left\langle \phi_0(w,\bar{w})\, \phi_1(w_1,\bar{w}_1)\, \ldots\, \phi_N(w_N,\bar{w}_N) \right\rangle ,$$

and suitable differential operators.

2.) **Energy momentum tensor as a descendant field**   2 pt

a) Express the holomorphic energy momentum tensor $T(z)$ as a descendant field of the identity operator **1**.

b) Use the expression for correlation functions of primaries with a descendant field to rederive the Conformal Ward Identity for primary fields.
Hint: The special correlator of a descendant discussed in the lecture is sufficient.

3.) **Operator algebra of primary fields**   6 pt
Let us consider the generic form of the operator algebra of two primary fields

$$\phi_k(z,\bar{z})\phi_\ell(0,0) \;=\; \sum_{\substack{s\\ \text{primaries}}} \sum_{\{\vec{k}\}} \sum_{\{\vec{\bar{k}}\}} C_{k\ell}^s \beta_{k\ell}^{s\{\vec{k}\}} \beta_{k\ell}^{s\{\vec{\bar{k}}\}} z^{h_s - h_k - h_\ell + |\vec{k}|}\, \bar{z}^{\bar{h}_s - \bar{h}_k - \bar{h}_\ell + |\vec{\bar{k}}|} \phi_s^{\{\vec{k}\}\{\vec{\bar{k}}\}}(0,0) .$$

For simplicity we assume in the following that the primaries ϕ_k and ϕ_ℓ have conformal weights $h_k = h_\ell = h$.

a) With the help of the Virasoro algebra evaluate the commutators $[L_1, L_{-1}^2]$, $[L_2, L_{-1}^2]$, $[L_1, L_{-2}]$, and $[L_2, L_{-2}]$.

b) In the lecture we determined $\beta_{k\ell}^{s\{1\}} = \frac{1}{2}$. Using the results of a), show that the constants $\beta_{k\ell}^{s\{1,1\}}$ and $\beta_{k\ell}^{s\{2\}}$ are given by

$$\beta_{k\ell}^{s\{1,1\}} \;=\; \frac{c - 12h - 4h_s + ch_s + 8h_s^2}{4(c - 10h_s + 2ch_s + 16h_s^2)} , \qquad \beta_{k\ell}^{s\{2\}} \;=\; \frac{2h - h_s + 4hh_s + h_s^2}{c - 10h_s + 2ch_s + 16h_s^2} ,$$

in terms of the conformal weights h and h_s, and the central charge c.