Tutorial Conformal Field Theory — Sheet 8

1.) **Positive definite Gram matrices for $c > 1$ and $h \gg 0$** `5 pt`

We want to argue that for sufficiently large conformal weights h and for $c > 1$ the Gram matrices $M^{(K)}(h,c)$ are positive definite.

a) Show that the the entries of the Gram matrix obey the asymptotic behavior

$$\begin{aligned}
\langle h|\, L^{\dagger}_{-\{\vec{k}'\}} L_{-\{\vec{k}\}}\, |h\rangle &= \mathcal{O}(h^{\min\{\ell(\vec{k}'),\ell(\vec{k})\}}) \ , \\
\langle h|\, L^{\dagger}_{-\{\vec{k}\}} L_{-\{\vec{k}\}}\, |h\rangle &= c_{\vec{k}}\, h^{\ell(\vec{k})} + \mathcal{O}(h^{\ell(\vec{k})-1}) \quad \text{with} \quad c_{\vec{k}} > 0 \ , \\
\langle h|\, L^{\dagger}_{-\{\vec{k}'\}} L_{-\{\vec{k}\}}\, |h\rangle &= \mathcal{O}(h^{\ell(\vec{k})-1}) \quad \text{for} \quad \vec{k}' \neq \vec{k} \quad \text{and} \quad \ell(\vec{k}) = \ell(\vec{k}') \ ,
\end{aligned}$$

in terms of the operators $L_{-\{\vec{k}\}} \equiv L_{-k_1} \dots L_{-k_n}$ of the vectors $\vec{k}$ and $\vec{k}'$ of length $\ell(\vec{k}) = n$ and $\ell(\vec{k}') = n'$ with the usual ordering convention $k_1 \leq k_2 \leq \dots \leq k_n$ and $k'_1 \leq k'_2 \leq \dots \leq k'_n$.
Notation: $p(h) = \mathcal{O}(h^n)$ implies that the polynomial $p(h)$ is at most of degree n.

b) Exemplify for the Gram matrices $M^{(K)}(h,c)$ up to level $K = 4$ that the above conditions are sufficient to argue that the matrices $M^{(K)}$ are positive definite for h sufficiently large.

c) `Bonus Problem` `3 bonus pt`
Show that for any level K there is a sufficiently large h such that the Gram matrix $M^{(K)}(h,c)$ is positive definite as consequence of the asymptotic behavior of its entries.

2.) **Fusion rules of the $m = 6$ unitary minimal models**

a) **Conformal families** `1 pt`
Write down the field content of the unitary minimal model of central charge $c = \frac{6}{7}$ (for $m = 6$).

b) **Closed subset of conformal families** `4 + 1 bonus pt`
For the $m = 6$ unitary minimal model determine a subset of conformal families that is closed with respect to the fusion rules.
Note: The points in this exercise are determined by the number of conformal families in the presented subset. If you find a subset with more than four conformal families you get an additional bonus point.