Tutorial Conformal Field Theory — Sheet 9

1.) $N = 1$ **Supersymmetry of the Tricritical Ising Model** 4 pt

Let us consider the extended $N = 1$ supersymmetric unitary minimal model series with $c(m) = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right)$, $m = 2, 3, \ldots$, of the $N = 1$ super-Virasoro algebra . We focus on the first non-trivial $N = 1$ unitary minimal model for $m = 3$.

a) Observe that the $N = 1$ unitary minimal model for $m = 3$ also occurs in the ordinary unitary minimal model series and determine its field content.

b) Identify the supercurrent $G(z)$ from the spectrum of primary fields, and show that this (correctly normalized) supercurrent $G(z)$ indeed fulfills the $N = 1$ super-Virasoro algebra.

Remark: The minimal model discussed in this exercise is associated to the Tricritical Ising Model in statistical mechanics at criticality. The lattice model of the Tricritical Ising Model generalizes the Ising model by allowing for vacant lattice sites. It is remarkable that such a simple statistical mechanics model realizes supersymmetry!

2.) **Dual primary fields in unitary minimal models** 6 pt

We consider primary fields $\phi_{(p,q)}$ and $\phi_{(p',q')}$ in unitary minimal models. Such primary fields are said to be dual to each other if they satisfy the condition

$$\left[\oint dz\, \phi_{(p,q)}(z), \oint dw\, \phi_{(p',q')}(w)\right] = 0 \ .$$

a) Verify that primary fields whose operator product expansion contains a single conformal family ϕ_h of conformal weight h, i.e.,

$$\phi_{(p,q)}(z)\phi_{(p',q')}(w) \sim (z-w)^{-\Delta} \left(\phi_h(w) + a(z-w)\partial_w \phi_h(w) + \ldots\right) \ , \qquad (1)$$

where a is some constant and $\Delta = h_{p,q} + h_{p',q'} - h$, are dual to each other if $\Delta = 2$.

b) Find all pairs of primary fields in unitary minimal models that satisfy eq. (1) with $\Delta = 2$.
Hint: There are three such pairs.

c) We will now prove that operator product expansion (1) with $\Delta = 2$ gives all solutions to the duality condition. Consider first the case $\Delta = 3$. Argue that the duality requirement can be satisfied only if there exists a relation between the descendant fields $\phi_h^{\{-1,-1\}}$ and $\phi_h^{\{-2\}}$, which forces ϕ_h to be either $\phi_{(1,2)}$ or $\phi_{(2,1)}$. But show that this is incompatible with the operator product expansion (1) with $\Delta = 3$ and $\phi_{(p,q)}$ and $\phi_{(p',q')}$ being both primary fields. Use a similar argument to rule out $\Delta > 3$.

d) Determine the value of the constant a in the operator product expansion (1).