Tutorial Conformal Field Theory — Sheet 10

1.) **Poisson resummation formula** 10 pt

The Poisson resummation is an important tool to determine modular properties of special functions. In this exercise we use the Poisson resummation technique to determine the modular properties of the Dedekind's eta function $\eta(\tau)$.

a) Given a function $f(x)$ which approaches zero suitable fast for $x \to \pm\infty$ — i.e., to be precise we consider $f(x)$ to be a Schwartz function. Then the Fourier transformed function $\hat{f}(p)$ is given by

$$\hat{f}(p) \,=\, \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x p} dx \ .$$

Prove the Poisson resummation formula for a Schwartz function f

$$\sum_{n\in\mathbb{Z}} f(n) \,=\, \sum_{n\in\mathbb{Z}} \hat{f}(n) \ .$$

b) Use the Poisson resummation formula to derive a summation identity for the function

$$f(x) \,=\, e^{-ax^2+bx} \ , \qquad \mathrm{Re}(a) > 0 \ .$$

c) Use the Jacobi triple product identity

$$\prod_{n=1}^{+\infty}(1-q^n)(1+q^{n-1/2}y)(1+q^{n-1/2}y^{-1}) \,=\, \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} y^n \ , \qquad |q|<1\ ,\ y\neq 0 \ ,$$

to derive the infinite sum formula

$$\eta(q) \,=\, \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3}{2}\left(n-\frac{1}{6}\right)^2} \ ,$$

for the Dedekind's eta function

$$\eta(q) \,=\, q^{1/24}\prod_{n=1}^{+\infty}(1-q^n) \ , \qquad q \,=\, e^{2\pi i \tau} \ .$$

d) Apply the result of b) to determine the behavior of the Dedekind's eta function $\eta(\tau)$ with respect to modular transformations.
Hint: In order to find the modular formula $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$, split the infinite sum into three suitable infinite sums after Poisson resummation.