Physikalisches Institut
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Theoretische Physik

Exercise 1
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Exercises on Conformal Field Theory

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http://www.th.physik.uni-bonn.de/klemm/cftss16/

–Home Exercises–

Due on April 22nd, 2016

H 1.1 Anharmonic ratios *(5 points)*

We consider conformal invariant ratios in $\mathbb{R}^d$ with $d > 2$.

a) Given four distinct points $x_1, \ldots, x_4$ show that the two anharmonic ratios (cross ratios),

$$\frac{|x_1 - x_2| \cdot |x_3 - x_4|}{|x_1 - x_3| \cdot |x_2 - x_4|} , \qquad \frac{|x_1 - x_2| \cdot |x_3 - x_4|}{|x_1 - x_4| \cdot |x_2 - x_3|} , \tag{1}$$

are conformal invariants.

b) Given N distinct points $x_1, \ldots, x_N$, determine the number of distinct (not necessarily algebraically independent) cross ratios.

Hint: Write $x_{ij} = |x_i - x_j|$ (how many distinct x_{ij} are there?) and consider the monomial m given by

$$m(x_1, \ldots, x_N) = \prod_{1 \leq i < j \leq N} x_{ij}^{a_{ij}} . \tag{2}$$

Argue that conformal invariance of this monomial requires

$$\sum_{j=1}^{i-1} a_{ji} + \sum_{j=i+1}^{N} a_{ij} = 0 \quad \textit{for all} \quad i = 1 \ldots N . \tag{3}$$

c) Argue that there can be at most $dN - (d+2)(d+1)/2$ algebraically independent cross ratios.

Hint: Compare the number of generators in the conformal group to the number of degrees of freedom of N points.

H 1.2 Conformal transformations *(5 points)*

Let us examine the group of conformal transformations of the Euclidean space $\mathbb{R}^d$ for $d > 2$. To this end we first note that the Lie group $SO(1, d+1)$ acts naturally on $\mathbb{R}^{d+2}$

via multiplication and we equip $\mathbb{R}^{d+2}$ with the metric $\eta_{1,d+1} = \mathrm{diag}(-1, 1, \ldots, 1)$.[1] Now consider the map

$$\begin{aligned} \iota :\ & \mathbb{R}^d \to \mathbb{RP}^{d+1} , \\ & x^\mu \mapsto \left[\, \tfrac{1}{2}(1+|x|^2) \,:\, x^1 \,:\, \ldots \,:\, x^d \,:\, \tfrac{1}{2}(1-|x|^2) \,\right] \quad \text{with} \quad |x|^2 = \sum_{k=1}^{d} \left(x^k\right)^2 , \end{aligned} \tag{4}$$

which embeds Euclidean $\mathbb{R}^d$ in the $(d+1)$-dimensional projective space $\mathbb{RP}^{d+1}$. This is the space of lines in $\mathbb{R}^{d+2}$ that pass through the origin, which is obtained by identifiying two points $x, y \in \mathbb{R}^{d+2}$ if there is a number $\lambda \in \mathbb{R}^* = \mathbb{R} - \{0\}$ such $x = \lambda \cdot y$. More precisely: Define an equivalence relation $\sim$ on $\mathbb{R}^{d+2}$ by $x \sim y$ if $x = \lambda \cdot y$ for $\lambda \in \mathbb{R}^*$, then

$$\mathbb{RP}^{d+1} = \left(\mathbb{R}^{d+2} - \{0\}\right) / \sim \ . \tag{5}$$

Hence, elements of $\mathbb{RP}^{d+1}$ are equivalence classes $[x]$ of points $x \in \mathbb{R}^{d+2}$ and we find $[x] = [\lambda x]$ for $\lambda \in \mathbb{R}^*$. Moreover, we define the action of $A \in SO(1, d+1)$ on $[x] \in \mathbb{RP}^{d+1}$ by $A([x]) = [Ax]$. Together with the map ι we finally obtain an action of $SO(1, d+1)$ on $\mathbb{R}^d$ declared by $\iota(A(x)) = A([\iota(x)])$ for $x \in \mathbb{R}^d$.

a) Show that the (connected component with the identity of the) Lie group $SO(1, d+1)$ has the same number of generators as the conformal group of $\mathbb{R}^d$.

   Hint: Why do the dimensions of $SO(1, d+1)$ and $SO(d+2)$ agree?

b) Check that ι maps points in $\mathbb{R}^d$ onto the projective light cone of $\mathbb{RP}^{d+1}$, which is the set of all $[x] \in \mathbb{RP}^{d+1}$ such that $\eta_{1,d+1}(x, x) = 0$.

   Note: Since $\eta_{1,d+1}(x, x) = 0$ implies $\eta_{1,d+1}(\lambda x, \lambda x) = 0$ for all $\lambda \in \mathbb{R}^*$, this condition is well defined on $\mathbb{RP}^{d+1}$.

c) Demonstrate that the $SO(1, d+1)$ matrices

$$\begin{pmatrix} 1 & & \\ & \mathbf{\Lambda}_{d\times d} & \\ & & 1 \end{pmatrix} , \qquad \begin{pmatrix} \frac{1+r^2}{2r} & 0 & \frac{1-r^2}{2r} \\ 0 & \mathbf{1}_{d\times d} & 0 \\ \frac{1-r^2}{2r} & 0 & \frac{1+r^2}{2r} \end{pmatrix} ,$$
$$\begin{pmatrix} 1+\frac{a^2}{2} & -\vec{a} & \frac{a^2}{2} \\ -\vec{a} & \mathbf{1}_{d\times d} & -\vec{a} \\ -\frac{a^2}{2} & \vec{a} & 1-\frac{a^2}{2} \end{pmatrix} , \quad \begin{pmatrix} 1+\frac{b^2}{2} & -\vec{b} & -\frac{b^2}{2} \\ -\vec{b} & \mathbf{1}_{d\times d} & \vec{b} \\ \frac{b^2}{2} & -\vec{b} & 1-\frac{b^2}{2} \end{pmatrix} , \tag{6}$$

   with $\mathbf{\Lambda}_{d\times d} \in SO(d)$, $r > 0$ and $\vec{a}, \vec{b} \in \mathbb{R}^d$, map to rotations, dilatations, translations and special conformal transformations on $\mathbb{R}^d$, respectively.

We have thus shown that the conformal group of $\mathbb{R}^d$ is $SO(1, d+1)$.

[1] In general the isometry group of the metric $\eta_{p,q} = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1)$ with p times (-1) and q times $+1$ is denoted as $O(p, q)$, and $SO(p, q)$ is the subgroup of elements with determinant $+1$.