Physikalisches Institut
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Theoretische Physik

Exercise 2
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Exercises on Conformal Field Theory

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http://www.th.physik.uni-bonn.de/klemm/cftss16/

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Due on April 29th, 2016

H 2.1 Correlation functions of quasi-primary fields *(7 points)*

Let us consider correlation function of quasi-primary fields $\phi_k(\vec{x})$ with scaling dimension Δ_k in d dimensions ($d > 2$). We consider conformal invariant ratios in $\mathbb{R}^d$ with $d > 2$.

a) Show that the three-point correlations function is given by

$$\langle \phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\phi_3(\vec{x}_3)\rangle \;=\; \frac{C_{123}}{|\vec{x}_1-\vec{x}_2|^{\Delta_1+\Delta_2-\Delta_3}|\vec{x}_1-\vec{x}_3|^{\Delta_1+\Delta_3-\Delta_2}|\vec{x}_2-\vec{x}_3|^{\Delta_2+\Delta_3-\Delta_1}} \;, \tag{1}$$

with an undetermined real constant C_{123}.

b) Show that the four-point correlation function takes the form

$$\langle \phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\phi_3(\vec{x}_3)\phi_4(\vec{x}_4)\rangle \;=\; f(A_1,A_2)\prod_{1\le k<l\le 4}|\vec{x}_k-\vec{x}_l|^{\Delta/3-\Delta_k-\Delta_l} \;, \tag{2}$$

where $\Delta = \Delta_1 + \ldots + \Delta_4$ and f is an undetermined function of the two anharmonic ratios

$$A_1 \;=\; \frac{|\vec{x}_1-\vec{x}_2|\cdot|\vec{x}_3-\vec{x}_4|}{|\vec{x}_1-\vec{x}_3|\cdot|\vec{x}_2-\vec{x}_4|} \;, \qquad A_2 \;=\; \frac{|\vec{x}_1-\vec{x}_2|\cdot|\vec{x}_3-\vec{x}_4|}{|\vec{x}_1-\vec{x}_4|\cdot|\vec{x}_2-\vec{x}_3|} \;. \tag{3}$$

Note: The representation of the four-point function given on the right-hand side of eq. (2) is not unique. One could, for example, replace the function f by f/A_1 and absorb the factor A_1 into the product of $|\vec{x}_k - \vec{x}_l|$, thus changing their exponent.

H 2.2 Special conformal transformations *(3 points)*

The action of the conformal group on a field $\phi(x)$ is defined by specifying the eigenvalues of $\phi(0)$ under $L_{\mu\nu}$, D and K_μ, which generate Lorentz transformations, dilatations and special conformal transformations, respectively:

$$L_{\mu\nu}\,\phi(0) = S_{\mu\nu}\,\phi(0) \;, \quad D\,\phi(0) = -i\Delta\,\phi(0) \;, \quad K_\mu\,\phi(0) = -\kappa_\mu\,\phi(0) \;. \tag{4}$$

Now we use the generator of translations, P_μ, to write $\phi(x) = \exp(-ix\cdot P)\phi(0)$. Together with the conformal algebra

$$\begin{aligned}
&[D,P_\mu] = iP_\mu \;, \quad [D,K_\mu] = -iK_\mu \;, \quad [K_\mu,P_\nu] = 2i\left(\eta_{\mu\nu}D - L_{\mu\nu}\right) \;,\\
&[K_\rho,L_{\mu\nu}] = i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right) \;, \quad [P_\rho,L_{\mu\nu}] = i\left(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right) \;,\\
&[L_{\mu\nu},L_{\rho\sigma}] = i\left(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}\right) \;,
\end{aligned} \tag{5}$$

equation (4) thus defines the action of $L_{\mu\nu}$, D and K_μ on $\phi(x)$.

a) Use the Backer–Campbell–Hausdorff formula,

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \dots \ , \tag{6}$$

to deduce that

$$K_\mu \, \phi(x) = \big[- \kappa_\mu - 2ix_\mu \Delta - 2x^\nu S_{\mu\nu} + 2x_\mu (x \cdot P) - x^2 P_\mu \big] \phi(x) \ . \tag{7}$$