Exercises on Conformal Field Theory

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–Home Exercises–

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H 3.1 Scale invariance in momentum space *(5 points)*

We consider a theory that is scale invariant in d dimensions, i.e., the theory is invariant under translations, rotations, and dilatations. In momentum space, a correlation function of N scalar field $\phi_i(\vec{x}_i)$, $i = 1, \ldots, N$, is represented by the Fourier transform $C(\vec{k}_1, \ldots, \vec{k}_N)$ for which

$$\langle \phi_1(\vec{x}_1) \cdots \phi_N(\vec{x}_N) \rangle = \int \frac{d\vec{k}_1}{(2\pi)^d} \cdot \ldots \cdot \frac{d\vec{k}_{N-1}}{(2\pi)^d} e^{i(\vec{k}_1 \cdot \vec{x}_1 + \ldots + \vec{k}_N \cdot \vec{x}_N)} C(\vec{k}_1, \ldots, \vec{k}_N) \,, \tag{1}$$

where $-\vec{k}_N = \vec{k}_1 + \ldots + \vec{k}_{N-1}$ is fixed by momentum conservation (translation invariance).

Note: In all parts of this exercise you may assume the respective integrals to exist. Also, if we are asking for a proportionality, you do not need to calculate integrals that just give a number.

a) Show that scale invariance of the theory implies

$$C(\vec{k}_1, \ldots, \vec{k}_N) = \Lambda^{(N-1)d - \Delta_1 - \ldots - \Delta_N} C(\Lambda\vec{k}_1, \ldots, \Lambda\vec{k}_N) \,, \tag{2}$$

where Δ_i are the scaling dimensions of the fields ϕ_i.

b) Given the two point correlations $\langle \phi_1(\vec{x}_1)\phi_2(\vec{x}_2) \rangle \sim |\vec{x}_1 - \vec{x}_2|^{-(d-2+\eta)}$ (with the critical exponent η), show that the two-point correlator in momentum space is given by

$$C(\vec{k}, -\vec{k}) \sim \frac{1}{|\vec{k}|^{2-\eta}} \,. \tag{3}$$

c) Consider now the special case of a scale invariant theory in two dimensions, i.e., $d = 2$. Introducing an infrared cutoff at length scale L to make the integral converge, show that the two-point function in coordinate space is proportional to

$$G_L(r) = \int_{L^{-1}}^{+\infty} \frac{dk}{k^{1-\eta}} J_0(kr) \,, \tag{4}$$

where $r = |\vec{x}_1 - \vec{x}_2|$ and J_0 is the zeroth-order Bessel function of the first type. Explain how this expression is compatible with two-point correlation function discussed in the lecture.

H 3.2 Clustery property *(5 points)*

Consider a generic four-point function of quasi-primary fields $\phi_k(z_k, \bar{z}_k)$, $k = 1, 2, 3, 4$, in a two-dimensional conformal field theory:

$$\begin{aligned} \langle 4 \text{ pt} \rangle &= \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\phi_4(z_4, \bar{z}_4)\rangle \\ &= f(\eta, \bar{\eta}) \prod_{1\le k<l\le 4} (z_k - z_l)^{h/3-h_k-h_l} (\bar{z}_k - \bar{z}_l)^{\bar{h}/3-\bar{h}_k-\bar{h}_l} \end{aligned} \tag{5}$$

with $h_k > 0$ and $\bar{h}_k > 0$ for all k and

$$h = \sum_{k=1}^{4} h_k \ , \quad \bar{h} = \sum_{k=1}^{4} \bar{h}_k \ , \quad \eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \ . \tag{6}$$

We are interested in the behaviour of the four-point function in a limit where the four points are paired such that the two points in each pair are much closer to each other than the distance between the pairs. In particular, we want to show that the four-point function then behaves as the product of the two two-point function corresponding to the two pairs.

a) Show that the three possibilities of such a pairing correspond to $\eta \to 0, 1, \infty$.
   Hint: For one of the cases you may want to first consider the second cross ratio

$$\widetilde{\eta} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \tag{7}$$

   and then determine η in terms of $\widetilde{\eta}$.

Since these three cases can be treated similarly, we now for definiteness consider the case in which $z_1 \to z_2$ and $z_3 \to z_4$. This motivates the following:

b) Show that the four-point function can be written as

$$\begin{aligned} &\langle 4 \text{ pt} \rangle \\ &= f(\eta, \bar{\eta}) \cdot z_{12}^{-2h_{12}} z_{34}^{-2h_{34}} \left(\eta \cdot \widetilde{\eta}\right)^{\frac{h_{12}+h_{34}}{3}} \left(\frac{z_{23}z_{24}}{z_{13}z_{14}}\right)^{\Delta_{12}} \left(\frac{z_{24}z_{14}}{z_{13}z_{23}}\right)^{\Delta_{34}} \cdot \left(\begin{array}{c} \text{antiholo.} \\ \text{part} \end{array} \right) , \end{aligned} \tag{8}$$

   where we have introduced the short hand notation $z_{ij} = z_i - z_j$ as well as the quantities

$$h_{12} = \frac{h_1 + h_2}{2} \ , \quad h_{34} = \frac{h_3 + h_4}{2} \ , \quad \Delta_{12} = \frac{h_1 - h_2}{2} \ , \quad \Delta_{34} = \frac{h_3 - h_4}{2} \ . \tag{9}$$

c) What is the qualitative difference between the case in which $h_1 = h_2$, $h_3 = h_4$ and the other cases?