Physikalisches Institut
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Theoretische Physik

Exercise 4
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Exercises on Conformal Field Theory

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Due on May 13th, 2016

H 4.1 Global conformal transformations in two dimensions *(4 points)*

We want to show that the group $SL(2,\mathbb{C})/\mathbb{Z}_2$ of global conformal transformations in two dimensions — also called the projective special linear group $PSL(2,\mathbb{C})$ — is isomorphic to the restricted four-dimensional Lorentz group $SO_+(1,3)$. Recall that the restricted Lorentz group $SO_+(1,3)$ is the group of linear transformations on a four-vector x^μ that leaves the square of the norm $|x^\mu|^2 \equiv -(x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2$ invariant and that preserves both orientation of space and of time. Thus, $SO_+(1,3)$ is the component of the Lorentz group connected to the identity.

a) Let H_2 be the (real) vector space of hermitian 2×2 matrices. To any four-vector $x = x^\mu e_\mu$ in terms of the basis e_μ, $\mu = 0,\ldots,3$, of $\mathbb{R}^4$, we associate the hermitian 2×2-matrix $X = \sum_{\mu=0}^3 x^\mu\sigma_\mu$ with σ_0 the identity matrix and $\sigma_1,\sigma_2,\sigma_3$ the Pauli matrices. Show that the map
$$f:\ \mathbb{R}^4 \to H_2,\ x\mapsto X = \sum_{\mu=0}^{3} x^\mu\sigma_\mu\ ,$$
is a vector space isomorphism, i.e., it is a one-to-one map respecting the vector space structure. Construct the inverse map f^{-1}.

b) First, demonstrate that $|x^\mu|^2 = -\det X$. Then show that for any S in $SL(2,\mathbb{C})$ the transformation $X\mapsto S^\dagger X S$ is well-defined in H_2. Furthermore, show that such transformations leave the square of the norm invariant. Argue that any matrix S in $SL(2,\mathbb{C})$ arises from a restricted Lorentz transformation.

Hint: Use the fact that both $SL(2,\mathbb{C})$ and $SO_+(1,3)$ are connected, i.e, that any group element of $SL(2,\mathbb{C})$ and $SO_+(1,3)$ continuously deforms to the identity.

c) Analyze which $SL(2,\mathbb{C})$ transformations leave the matrix X invariant. Conclude that the group $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\mathbb{Z}_2$ is isomorphic to the restricted Lorentz group $SO_+(1,3)$.

d) Show that the Möbius transformations
$$z\mapsto \frac{az+b}{cz+d}\ ,\qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{C})\ ,$$
form a group, which is isomorphic to the restricted Lorentz group $SO_+(1,3)$.

H 4.2 Three-point correlator revisited *(6 points)*

In two dimensions we want to calculate the three-point correlator of three quasi-primary fields $\phi_i(z_i, \bar{z}_i)$ for $i = 1, 2, 3$ with conformal weight $(h_i, \bar{h}_i)$ using infinitesimal global conformal transformations.

a) For infinitesimal Möbius transformations, i.e.,

$$z \mapsto \frac{(1 + c_0/2)z + c_{-1}}{-c_1 z + (1 - c_0/2)} \ , \quad c_{-1}, c_0, c_1 \text{ infinitesimal} \ ,$$

determine the variation $\epsilon(z)$ of the infinitesimal global conformal transformation

$$z \mapsto \tilde{z} = z + \epsilon(z) \ .$$

b) Use the infinitesimal transformation law for global conformal transformations acting on (quasi-)primary fields — i.e.,

$$\delta_{\epsilon,\bar{\epsilon}} \phi(z, \bar{z}) = - \left[h(\partial_z \epsilon(z)) + \epsilon(z)\partial_z + \bar{h}(\partial_{\bar{z}} \bar{\epsilon}(\bar{z})) + \bar{\epsilon}(\bar{z})\partial_{\bar{z}} \right] \phi(z, \bar{z}) \ ,$$

with $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ an infinitesimal and a conjugate infinitesimal Möbius transformation, respectively — to derive differential equations for the three-point correlator

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3) \rangle \ .$$

Here, ϕ_i, $i = 1, 2, 3$, are quasi-primary fields with conformal weight $(h_i, \bar{h}_i)$.

c) Use the derived differential equations to derive the three-point correlator:

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3) \rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{13}^{h_1+h_3-h_2} z_{23}^{h_2+h_3-h_1} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_2} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}}$$

with $z_{ij} = z_i - z_j$ and $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$.

Hint: Determine the three-point correlator in three steps: First show that the holomorphic part of the correlator is a function of z_{12} and z_{13} only. In the next step constrain this function further to the form $g(z_{12}/z_{13}) z_{12}^{-h_1-h_2-h_3}$. Finally, determine the function $g(z_{12}/z_{13})$, which allows you to derive the anticipated result for (the holomorphic part of) the correlator.