Exercises on Conformal Field Theory

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–Home Exercises–

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H 5.1 Primary fields of the free boson CFT *(7 points)*

We consider the two-dimensional conformal field theory of a free boson given in terms of the action

$$S[\phi] = 2g \int dz d\bar{z}\, \partial_z \phi(z, \bar{z}) \partial_{\bar{z}} \phi(z, \bar{z}) \ .$$

Recall that the holomorphic energy momentum tensor $T(z)$ of the free boson conformal field theory is given by

$$T(z) = -2\pi g : \partial\phi(z)\partial\phi(z) : ,$$

Here, $\partial\phi$ is the chiral primary field of conformal weight $(h, \bar{h}) = (1, 0)$ with the operator product expansion

$$\mathcal{R}(\partial_z \phi(z, \bar{z}) \partial_w \phi(w, \bar{w})) \sim -\frac{1}{4\pi g} \frac{1}{(z-w)^2} \ .$$

Now we want to study the spectrum of primary fields of the free boson conformal field theory.

a) Show that the normal-ordered operators $\mathcal{V}_\alpha \ =: e^{i\alpha\phi(z,\bar{z})} :$ are primary fields and determine their conformal weights h and $\bar{h}$.

Hint: Determine operator product expansion with the energy momentum tensor $T(z)$.

b) Consider two operators A and B linear in the creation and annihilation operators $a^\dagger$ and a of the harmonic oscillators, i.e., $[a^\dagger, a] = 1$. Show that these operators obey the relation

$$: e^A :: e^B := : e^{A+B} : \ e^{\langle AB \rangle} \ ,$$

with $\langle AB \rangle = \langle 0|AB|0 \rangle$.

Hint: Use the Baker–Campbell–Haussdorff formula

$$e^{-A} B e^A = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \dots \ .$$

c) As the free bosonic field can be seen as a collection of decoupled harmonic oscillators argue that the two-point correlation function is given by

$$\langle \mathcal{V}_\alpha(z, \bar{z}) \mathcal{V}_\beta(w, \bar{w}) \rangle = \begin{cases} |z-w|^{-\frac{\alpha^2}{2\pi g}} & \text{for } \alpha = -\beta \\ 0 & \text{else} \end{cases} \ .$$

Hint: Derive the leading term in the operator product expansion $\mathcal{R}(\mathcal{V}_\alpha(z, \bar{z}) \mathcal{V}_\beta(w, \bar{w}))$, and use that $\mathcal{V}_0(z, \bar{z})$ is the identity operator.

H 5.2 The free Dirac fermion and bosonization *(3 points)*

Consider the free Dirac (complex) fermion $\underline{\chi} = (\bar{\chi}, \chi)$ in two dimensions, which is given in terms of two Majorana (real) fermions $\Psi_1 = (\bar{\psi}_1, \psi_1)$ and $\Psi_2 = (\bar{\psi}_2, \psi_2)$, i.e.,

$$\underline{\chi} = \frac{\sqrt{2}}{2}\left(\Psi_1 + i\Psi_2\right) \ , \qquad \underline{\chi}^* = \frac{\sqrt{2}}{2}\left(\Psi_1 - i\Psi_2\right)$$

and the action

$$S[\underline{\chi}] \ = \ 2g \int d^2 z \left(\chi^* \partial_{\bar{z}} \chi + \bar{\chi}^* \partial_z \bar{\chi}\right) \ ,$$

a) Calculate the two-point correlation function of the Dirac fermion and the holomorphic energy momentum tensor $T(z)$ of the free Dirac fermion theory.

b) Show that $c = 1$ is the central charge of the free Dirac fermion theory. This central charge indicates that the free Dirac fermion conformal field theory in two dimensions can actually be represented in terms of the free boson conformal field theory. This identification is called bosonization. With regard to the previous exercise identify the primary field in the free boson conformal field theory that represents the complex fermion in the Dirac fermion theory.

   Hint: Construct appropriate vertex operators from the chiral part $\varphi(z)$ of the free boson $\phi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z})$.