Physikalisches Institut
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Theoretische Physik

Exercise 9
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Exercises on Conformal Field Theory

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Due on June 24th, 2016

H 9.1 Positive definiteness of the Gram matrices for $c > 1$ and $h \gg 0$ *(6 points)*
We want to argue that for sufficiently large conformal weights h and for $c > 1$ the Gram matrices $M^{(K)}(h,c)$ are positive definite.

a) Show that the the entries of the Gram matrix obey the asymptotic behavior

$$\begin{aligned}
\langle h|\, L^\dagger_{-\{\vec{k}\}} L_{-\{\vec{k}\}}\, |h\rangle \;&=\; c_{\vec{k}}\, h^{\ell(\vec{k})} + \mathcal{O}(h^{\ell(\vec{k})-1}) \quad \text{with} \quad c_{\vec{k}} > 0 \;, \\
\langle h|\, L^\dagger_{-\{\vec{k}'\}} L_{-\{\vec{k}\}}\, |h\rangle \;&=\; \mathcal{O}(h^{\min\{\ell(\vec{k}'),\ell(\vec{k})\}}) \;, \\
\langle h|\, L^\dagger_{-\{\vec{k}'\}} L_{-\{\vec{k}\}}\, |h\rangle \;&=\; \mathcal{O}(h^{\ell(\vec{k})-1}) \quad \text{for} \quad \vec{k}' \neq \vec{k} \quad \text{and} \quad \ell(\vec{k}) = \ell(\vec{k}') \;,
\end{aligned}$$

in terms of the operators $L_{-\{\vec{k}\}} \equiv L_{-k_1} \dots L_{-k_n}$ with the vectors $\vec{k}$ and $\vec{k}'$ of length $\ell(\vec{k}) = n$ and $\ell(\vec{k}') = n'$ with the usual ordering convention $k_1 \le k_2 \le \dots \le k_n$ and $k'_1 \le k'_2 \le \dots \le k'_n$. *(3 points)*
Notation: $p(h) = \mathcal{O}(h^n)$ denotes that the polynomial $p(h)$ is at most of degree n.

b) Exemplify for the Gram matrix $M^{(4)}(h,c)$ at level $K = 4$ that the above conditions are sufficient to argue that the matrix $M^{(4)}$ is positive definite for sufficiently large conformal weight h. *(2 points)*
Hint: Use Sylvester's criterion for positive definite Hermitian matrices, i.e., a Hermitian matrix M of dimension $n \times n$ is positive definite, if and only if all leading principal minors m_k with $k = 1, \dots, n$ are positive. The leading principal minors m_k are the determinants of the $k \times k$ upper left sub-matrices of M.

c) Show that for any level K there is a sufficiently large h such that the Gram matrix $M^{(K)}(h,c)$ is positive definite as a consequence of the asymptotic behavior of its entries. *(1 point)*

H 9.2 Fusion rules of the tricritical Ising model *(4 points)*

The unitary minimal model with $c = \frac{7}{10}$ $(m = 4)$ can be identified as the tricritical Ising model. We want to determine its conformal families, their fusion rules, and identify a subset of conformal families that closes with respect to the fusion rules.

a) **Conformal families and fusion rules** *(3 points)*
Write down the field content of the unitary minimal model of central charge $c = \frac{7}{10}$ $(m = 4)$. Further, list all fusion rules.

b) **Closed subset of conformal families** *(1 point)*
Determine a (proper and non-trivial) subset of conformal families that is closed with respect to the fusion rules that you found in the previous item.