Physikalisches Institut
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Theoretische Physik

Exercise 10
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Exercises on Conformal Field Theory

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http://www.th.physik.uni-bonn.de/klemm/cftss16/

–Home Exercises–

Due on July 1st, 2016

H 10.1 Selection rules for structure constants *(4 points)*

In this exercise we want to derive selection rules for the structure constants of a conformal field theory by employing singular vectors. In particular, consider a primary ϕ_0 — whose corresponding highest weight state is denoted $|h_0\rangle$ — with a null descendant $|\chi_N\rangle$

$$|\chi_N\rangle = \sum \alpha_{\{\vec{k}\}} L_{-\{\vec{k}\}} |h_0\rangle \tag{1}$$

at level N. Here, $L_{-\{\vec{k}\}} \equiv L_{-k_1} \ldots L_{-k_l}$ with the usual ordering convention $k_1 \leq k_2 \leq \cdots \leq k_l$ and the sum is over all $\vec{k}$ such that $\sum k_i = N$. This null descendant enables us determine when the three-point correlator

$$\langle \phi_0(z_0)\phi_1(z_1)\phi_2(z_2) \rangle \tag{2}$$

of the primaries ϕ_0, ϕ_1, ϕ_2 necessarily has to vanish. For simplicity we supress any antiholomorphic dependence.

a) Use the null descendant to write down a differential equation for the above three-point correlator. Then, use the explicit form of the three-point function and consider the leading term the limit $z_0 \to z_1$. Deduce a polynomial equation for the conformal weights h_0, h_1, h_2.

b) Explain how this equation can be used to check whether a given three-point function has to vanish. Check explicitly that the fusion rules of the $m = 3$ minimal model are in accord with the derived selection rules.

H 10.2 Differential equation for a four-point function *(6 points)*

Let us consider the $m = 3$ minimal model, which has a primary field σ with $h_\sigma = 1/16$ (for simplicity we again neglect any antiholomorphic dependence). Recall that the corresponding primary state $|h_\sigma\rangle$ has a null descendant

$$|\chi_{1,2}\rangle = \left(L_{-2} - \frac{4}{3} L_{-1}^2 \right) |h_\sigma\rangle \tag{3}$$

at level 2. In this exercise we will see how this null state puts further constraints on the four-point function

$$\langle \sigma(z_0)\sigma(z_1)\sigma(z_2)\sigma(z_3) \rangle = \prod_{0 \leq i < j \leq 3} z_{ij}^{-\frac{1}{24}} f(\eta) \ , \tag{4}$$

where $z_{ij} = z_i - z_j$ and $\eta = z_{01}z_{23}/(z_{03}z_{21})$.

a) Show that

$$\left(\sum_{k=1}^{3}\left[\frac{1}{16z_{0k}^2}+\frac{1}{z_{0k}}\frac{d}{dz_k}\right]-\frac{4}{3}\frac{d^2}{dz_0^2}\right)\langle\sigma(z_0)\sigma(z_1)\sigma(z_2)\sigma(z_3)\rangle = 0\ . \tag{5}$$

b) Note that the derivates in eq. (5) act on the prefactors $z_{ij}^{-\frac{1}{24}}$ and, via chain rule, on $f(\eta)$. This effectively leads to the replacements

$$\begin{aligned}\frac{d}{dz_k} &= a_k(\vec{z}) + b_k(\vec{z})\frac{d}{d\eta}\ , \qquad k = 1,\dots,3\\ \frac{d^2}{dz_0^2} &= c(\vec{z}) + d(\vec{z})\frac{d}{d\eta} + e(\vec{z})\frac{d^2}{d\eta^2}\end{aligned} \tag{6}$$

for some functions a_k, b_k, c, d, and e. Once you have obtained these expressions take the limits $z_1 \to 0, z_2 \to 1, z_3 \to \infty, z_0 \to \eta$ to arrive at

$$\begin{aligned}\frac{d}{dz_1} &= +\frac{1}{24}\left(\frac{1}{\eta}+1\right) + (\eta-1)\frac{d}{d\eta}\ ,\\ \frac{d}{dz_2} &= +\frac{1}{24}\left(\frac{1}{\eta-1}-1\right) - \eta\frac{d}{d\eta}\ ,\\ \frac{d}{dz_3} &= 0\ ,\\ \frac{d^2}{dz_0^2} &= \frac{25}{576}\left(\frac{1}{\eta^2}+\frac{1}{(\eta-1)^2}\right) + \frac{1}{288}\frac{1}{\eta(\eta-1)} - \frac{1}{12}\left(\frac{1}{\eta}+\frac{1}{\eta-1}\right)\frac{d}{d\eta} + \frac{d^2}{d\eta^2}\ .\end{aligned} \tag{7}$$

Note that in this formulation $d/d\eta$ acts on $f(\eta)$ only, but not on the z_{ij}.

c) Show that $f(\eta)$ is annihilated by the differential operator

$$\frac{d^2}{d\eta^2}+\left[\frac{3(2\eta-1)}{4\eta(\eta-1)}-\frac{1}{12}\left(\frac{1}{\eta}+\frac{1}{\eta-1}\right)\right]\frac{d}{d\eta}-\frac{5}{144}\left[\frac{1}{\eta^2}+\frac{1}{(\eta-1)^2}\right]+\frac{5}{144\eta(\eta-1)}\ . \tag{8}$$