Physikalisches Institut
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Theoretische Physik

Exercise 11
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Exercises on Conformal Field Theory

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http://www.th.physik.uni-bonn.de/klemm/cftss16/

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Due on July 8th, 2016

H 11.1 Modular invariance of the Ising model at criticality *(10 points)*

In this exercise we want to show that the modular invariance of the Ising model. Our first task is to derive the modular properties of the minimal model characters, which are given by

$$\chi^{(m)}_{r,s}(\tau) \;=\; K^{(m)}_{r,s}(q) - K^{(m)}_{r,-s}(q) \;, \qquad K^{(m)}_{r,s}(q) \;=\; \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{\frac{(N\,n+\lambda(r,s))^2}{2N}} \;, \tag{1}$$

in terms of

$$q = e^{2\pi i\tau} \;, \quad N = 2m(m+1) \;, \quad \lambda(r,s) = r(m+1) - s\,m \;,$$

and the Dedekind's η-function

$$\eta(q) \;=\; q^{1/24} \prod_{n=1}^{+\infty} (1-q^n) \;. \tag{2}$$

Recall that the η-function has the modular properties

$$T: \;\eta(\tau+1) = e^{\frac{2\pi i}{24}}\eta(\tau) \;, \qquad S: \;\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau) \;. \tag{3}$$

a) Given a function $f(x)$ which approaches zero suitable fast for $x \to \pm\infty$, i.e., — to be precise — we consider $f(x)$ to be a Schwartz function. Then the Fourier transformed function $\hat{f}(p)$ is given by

$$\hat{f}(p) \;=\; \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x p} dx \;.$$

Prove the Poisson resummation formula for a Schwartz function f

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; \sum_{n\in\mathbb{Z}} \hat{f}(n) \;.$$

Hint: Use the Fourier mode expansion of the periodic function $\widehat{F}(x) = \widehat{F}(x+1)$ given by $\hat{F}(x) = \sum_{n\in\mathbb{Z}} f(x+n)$.

b) Use the Poisson resummation formula to derive a summation identity for the function

$$f(x) \;=\; e^{-ax^2+bx} \;, \qquad \mathrm{Re}(a) > 0 \;.$$

c) Use the modular properties (3) of the η function to show that

$$T : \chi_{r,s}^{(m)}(\tau + 1) = e^{2\pi i\left(h_{r,s}(m) - \frac{c(m)}{24}\right)} \chi_{r,s}^{(m)}(\tau) \ .$$

with

$$h_{r,s}(m) = \frac{\lambda(r,s)^2 - 1}{2N} \ , \quad c(m) = 1 - \frac{12}{N} \ .$$

d) Now we want to determine the transformation of the minimal model characters $\chi_{r,s}^{(m)}(\tau)$ under the S transformation, which we will do in two steps:

(i) First use eq. (3) and the result of b) to obtain

$$K_{r,s}^{(m)}(-1/\tau) = \frac{1}{\sqrt{N}\eta(\tau)} \sum_{\mu=0}^{N-1} \sum_{k \in \mathbb{Z}} e^{\frac{2\pi i \lambda(r,s)\mu}{N}} q^{\frac{(Nk+\mu)^2}{2N}} \ . \tag{4}$$

(ii) Second, show that for the minimal model character

$$\chi_{r,s}^{(m)}(-1/\tau) = \frac{4}{\sqrt{N}\eta(\tau)} \sum_{\mu \in A} \sum_{k \in \mathbb{Z}} \sin\left(\frac{\pi\, r\, \mu}{m}\right) \sin\left(\frac{\pi\, s\, \mu}{m+1}\right) q^{\frac{(Nk+\mu)^2}{2N}} \ , \tag{5}$$

where the summation of μ has been reduced to the set

$$A = \left\{\mu = 1, \ldots, m(m+1) - 1 \ \middle| \ m \nmid \mu \text{ and } (m+1) \nmid \mu \right\} \ . \tag{6}$$

Hint: Use the trigonometric identity $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

e) Let us now consider the $m = 3$ minimal model. Use the previous equation to find the transformations

$$\begin{aligned}
\chi_{1,1}^{(3)}(-1/\tau) &= \frac{\chi_{1,1}^{(3)}(\tau) + \chi_{2,1}^{(3)}(\tau) + \sqrt{2}\chi_{2,2}^{(3)}(\tau)}{2} \ , \\
\chi_{2,1}^{(3)}(-1/\tau) &= \frac{\chi_{1,1}^{(3)}(\tau) + \chi_{2,1}^{(3)}(\tau) - \sqrt{2}\chi_{2,2}^{(3)}(\tau)}{2} \ , \\
\chi_{1,1}^{(3)}(-1/\tau) &= \frac{\chi_{1,1}^{(3)}(\tau) - \chi_{2,1}^{(3)}(\tau)}{\sqrt{2}} \ .
\end{aligned} \tag{7}$$

f) Determine the partition function $Z_{\text{Ising}}(\tau)$ of the Ising model, and show with c) and e) that it is modular invariant.