Tutorial Dualities in Field and String Theory — Sheet 1

1.) **Harmonic oscillator** `2 pt`

Show that the spectrum (the stationary eigenvalue wave functions) of the quantum mechanical harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\,\omega^2 \hat{x}^2 \tag{1}$$

is invariant under the duality transformation

$$D: \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\hat{p}}{m\omega} \\ -m\omega\hat{x} \end{pmatrix} \tag{2}$$

2.) **Ising model**

The ferromagnetic Ising model on a square $N \times M$ lattice (with periodic boundary conditions) has the partition function

$$Z_{\text{Ising}} = \sum_{\{\sigma\}} \exp\left(-\beta \sum_{\langle ij\rangle} E_{ij}\right) , \quad \beta = \frac{1}{k_B T} , \quad E_{ij} = -J\sigma_i\sigma_j , \quad \sigma_i \in \{-1, +1\} , \quad J > 0$$

The first sum is taken over all spin configurations $\{\sigma\}$ of the set of spin variables σ_i at each lattice site i, whereas the sum in the exponent runs over all links $\langle ij\rangle$ between nearest neighbor lattice sites i and j (nearest neighbor interaction).

a) Low temperature expansion: `2 pt`

Show that the low temperature phase ($\beta J \gg 0$) of the partition function is given by:

$$Z_{\text{low}} = 2\, e^{2NM\,\beta J} \sum_{\{\text{loops}\}} e^{-2\beta J\,(\text{length loop})} \tag{3}$$

Here the sum runs over all possible (not necessarily connected) loops enclosing lattice domains. (These are loops in the dual square lattice.)

Hint: Consider all possible perturbations of a ferromagnetic ground state.

b) Show that `2 pt`

$$Z_{\text{Ising}} = \sum_{\{\sigma\}} \prod_{\langle ij\rangle} \left(\cosh(\beta J) + \sigma_i\sigma_j \sinh(\beta J)\right) \tag{4}$$

c) High temperature expansion: `2 pt`

Show that the high temperature phase ($\frac{1}{\beta J} \gg 0$) of the partition function is given by:

$$Z_{\text{high}} = 2^{NM} \cosh(\beta J)^{2NM} \sum_{\{\text{loops}\}} \tanh(\beta J)^{(\text{length loop})} \tag{5}$$

Here the sum runs over all possible (not necessarily connected) loops in the square lattice.

Hint: Use the partition function (4)*.*

d) `2 pt`

Demonstrate that in the thermodynamic limit $N, M \to \infty$ eq. (3) is a valid expansion at low temperatures but breaks down at high temperatures. Analogously, argue that in the thermodynamic limit $N, M \to \infty$ eq. (5) is a good expansion at high temperatures but breaks down at low temperatures.