Tutorial Dualities in Field and String Theory — Sheet 2

1.) **Dirac-Schwinger-Zwanziger quantization** 4 pt

a) Consider a point electric charge e and a point magnetic charge g and compute the field angular momentum

$$\vec{L} = \int d^3x \, \vec{x} \times (\vec{E} \times \vec{B}) \ .$$

Show that $\vec{L}$ is well-defined and independent of the distance between e and g.

b) Show that – demanding that the angular momentum be quantized in units $\frac{\hbar}{2}$ – yields the Dirac quantization condition:

$$e\, g = 2\pi\hbar N \text{ with } N \in \mathbb{Z}$$

c) Prove the Dirac-Schwinger-Zwanziger quantization condition

$$e\, g' - e'\, g = 2\pi\hbar N \text{ with } N \in \mathbb{Z}$$

for two dyonic point charges (e, g) and (e', g') with combined electric and magnetic charges.

d) Explore the allowed solutions to the Dirac-Schwinger-Zwanziger quantization condition assuming the existence of an electron with charge $(e, 0)$. Recall that under CP transformations the electric and magnetic charges are mapped according to $(e, g) \to (-e, g)$. Show that there are solutions, which lead to a CP violating spectrum. Show that there are solutions furnishing a CP invariant spectrum, which contains dyons carrying half of the electric charge of an electron.

2.) **ϕ^4 theory in two space-time dimensions** 2 pt

Calculate the (non-trivial) static finite-energy solitonic solutions $\phi(x)$ for the ϕ^4-theory of a real scalar field $\phi(x)$ in two space-time dimensions

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - U(\phi) \quad \text{with} \quad U(\phi) = \frac{\lambda}{2}\left(\phi^2 - a^2\right)^2 \quad \lambda > 0 \ .$$

Determine the energy and the topological charge of this solitonic solution. Does it saturate the Bogomol'nyi bound?

3.) **Energy of solitonic solutions in two space-time dimensions** 4 pt

Consider a real scalar field in two space-time dimensions with Lorentz invariant Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)$. Let $\phi_{x_0,v}(x,t)$ be a 2-parameter family of (dynamical) solitonic solutions to the equations of motion with finite energy and fixed boundary conditions $\lim_{x\to\pm\infty}\phi_{x_0,v}(x,t) = \phi_\pm$. Here $\phi_+ \neq \phi_-$ are isolated, nearby global minima of the potential with $U(\phi_\pm) = 0$. The two parameters of the family are the position x_0 of the center of energy and its velocity v fixed relative to a reference frame. Show that the energy of the configuration statisfies the usual relations

$$E_v = \frac{E_0}{\sqrt{1-v^2}} \ , \qquad E^2 = P^2 + M^2 \ .$$