Tutorial Dualities in Field and String Theory — Sheet 3

1.) **Soliton/anti-Soliton force – ϕ^4 theory in two space-time dimensions** 6 pt

a) Consider a d-dimensional Lagrangian $\mathcal{L}$ invariant under translations $x^\mu \to x^\mu + \epsilon^\mu$. The conserved Noether currents associated with the translation symmetry are comprised in the energy momentum tensor and its components are denoted by T^μ_ν. Compute the components of the energy momentum tensors and verify that the Noether charges take the form:

$$E = \int d^d x \, T^0_0 = \int d^d x \, (\frac{\delta \mathcal{L}}{\delta(\partial_0 \phi)} \, \partial_0 \phi - \mathcal{L}) \, ,$$
$$P_i = -\int d^d x \, T^0_i = -\int d^d x \, \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi)} \, \partial_i \phi \, .$$

Apply the result to determine the currents T^0_μ for a scalar theory with Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U(\phi)$.

b) Compute the approximate interaction energy between a well-separated pair of (static) solitonic and anti-solitonic solutions in the ϕ^4 theory in two space-time dimensions. As one might expect the force will turn out to be attractive. The Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - U(\phi) \quad \text{with} \quad U(\phi) = \frac{\lambda}{2}\left(\phi^2 - a^2\right)^2$$

To find the interaction energy make an appropriate superposition ansatz for the soliton/anti-soliton pair. The ansatz needs to solve the equation of motion only approximately, as the the two solitons are assumed to be very well separated. Then compute the force as the time derivative of the momentum of this soliton/anti-soliton field configuration for large separations.
Hint: Use the solitonic solution of exercise 2/sheet 2. Use an appropriate cut-off for computing the force.

2.) **Solitons in the Sine-Gordon Theory** 4 pt

a) Show that

$$s^{(2)}_{++}(x,t) = \frac{4}{\beta} \arctan\left(\frac{v \sinh(\sqrt{\alpha}\,\gamma x)}{\cosh(\sqrt{\alpha}\,\gamma v t)}\right) \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1-v^2}} \, ,$$

is a (dynamical) solution to the Sine-Gordon Theory:

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - U(\phi) \quad \text{with} \quad U(\phi) = \frac{\alpha}{\beta^2}\left(1 - \cos\beta\phi\right)$$

Calculate the topological charge Q and the energy E of the solution $s^{(2)}_{++}$, and compare with the energy and topological charge of the static solitonic solution.

b) Compute the asymptotic behavior of the solution $s^{(2)}_{++}$ for $t \to \pm\infty$. Show that it decomposes into a sum of two solitonic solutions and a vacuum solution. Interpret the parameters in the result and study the limit $v \to 0$.

c) Illustrate and describe the dynamics of the solution $s^{(2)}_{++}$ by sketching its energy density for suitable times.