Tutorial Dualities in Field and String Theory — Sheet 4

1.) **Instantons** 2 pt

Consider the D-dimensional Euclidean action

$$S[A] = \frac{1}{4}\int d^D x \, \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \ ,$$

of a (non-Abelian) gauge boson A_μ. Show that there can only be non-trivial stationary instanton solutions of $S[A]$ in $D = 4$ dimensions unless $S[A] = 0$.
Hint: Use similar scaling argument as in Derrick's theorem.

2.) **Properties of homotopy group** 3 pt

a) Given two spaces X and Y, show that the homotopy groups obey:

$$\pi_q(X \times Y) = \pi_q(X) \times \pi_q(Y)$$

b) Determine the fundamental group $\pi_1(S^1)$ of the circle S^1, and determine the fundamental group $\pi_1(T^n)$ of the n-torus T^n.

c) Given an example of a two-dimensional topological space with a non-Abelian fundamental group.

d) Show that the groups $\pi_q(X)$ for any topological space X are Abelian for $q > 1$.
Hint: Use the graphical group operation of q-cubes I^q discussed in the lecture to argue that the multiplication of two group elements commutes for $q > 1$.

3.) **Skyrmions of the $SU(2) \times SU(2)$ pion model** 2 pt

We consider in four space-time dimensions the $SU(2) \times SU(2)$ pion model spontaneously broken to the diagonal $SU(2)_D$. We want to show that this theory has Skyrmion solutions that are topologically classified by $\pi_3((SU(2) \times SU(2))/SU(2)_D) = \mathbb{Z}$.

a) Show that the homogeneous space $(SU(2) \times SU(2))/SU(2)_D$ is naturally identified with the Lie group $SU(2)$ with the identity element e of the Lie group $SU(2)$ as a canonical marked point. Which elements of $SU(2) \times SU(2)$ correspond to this marked point in the coset $(SU(2) \times SU(2))/SU(2)_D$?

b) Show that the Lie group $SU(2)$ is topologically the three sphere S^3. Then infer with the help of $\pi_3(S^3) = \mathbb{Z}$ the topological classification of Skyrmions in the $SU(2) \times SU(2)$ pion model.

4.) **Hopf fibration** `3 pt`

We want to show that there exist interesting solutions in $D = 3$ for a field ϕ taking values in S^2, classified by $\pi_3(S^2) = \mathbb{Z}$.

a) A *fibration* is a map $\pi : X \to B$ satisfying the *homotopy lifting property*: Given a map $f : Y \to X$ from any topological space Y into X and given a map $\bar{g} : Y \to B$ homotopic to $\bar{f} = \pi \circ f : Y \to B$, then there exists a map g homotopic to f such that $\bar{g} = \pi \circ g$.
In such a fibration, the space B is called the *base space* of the fibration. For a base space with marked point $*$ the space $\pi^{-1}(*)$ is called the *fiber* F, whereas for any x in B, $F_x = \pi^{-1}(x)$ is called the *fiber over* x. With this terminology the fibration is conveniently written as $F \hookrightarrow X \xrightarrow{\pi} B$.
Derive an explicit expression for the so-called *Hopf fibration* $S^1 \to S^3 \xrightarrow{\pi} S^2$ starting from the representation $S^3 : \{(z_1, z_2) \in \mathbb{C} : |z_1|^2 + |z_2|^2 = 1\}$.
Hint: Consider a projection π that forgets an (obvious) $U(1)$ factor.

b) For a fibration (assuming B is path connected), one has a long exact sequence of homotopy groups

$$\ldots \to \pi_n(F) \to \pi_n(X) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots$$

called the *exact homotopy sequence*. Recall that an exact sequence of groups is a concatenation of group homomorphisms such that the image of each map is the kernel of the subsequent map.
Starting form the Hopf fibration $S^1 \to S^3 \to S^2$, show that $\pi_3(S^2) = \pi_2(S^2) = \mathbb{Z}$ (using $\pi_3(S^3) = \mathbb{Z}$).