Tutorial Dualities in Field and String Theory — Sheet 5

1.) **Monopole configurations** `3 pt`

Consider the $SU(2)$ Georgi-Glashow model discussed in the lecture.

a) If $\phi^a \to v\frac{x^a}{r}$ as $r \to +\infty$ (with $r = \sqrt{\sum_a |x^a|^2}$), show that

$$n_m \;=\; \frac{1}{8\pi v^3}\int_{S^2_\infty} dS^i\, \epsilon^{ijk}\epsilon^{abc}\phi^a\partial^j\phi^b\partial^k\phi^c \;=\; 1$$

b) Construct a map $S^2_\infty \to S^2$ with arbitrary integer winding number n_m.

2.) **Lower bound on the 't Hooft-Polyakov monopole mass** `1 pt`

The Bogomol'nyi bound for the mass of a 't Hooft-Polyakov monopole is given by:

$$M_M \geq v|g|$$

Estimate a lower bound on the magnetic monopole mass M_M by assuming a $W_\pm$-boson mass $M_W \simeq 85\,\text{GeV}$ and with the fine structure constant $\alpha = \frac{1}{137}$.

3.) $SL(2,\mathbb{Z})$ **invariance of the BPS-mass formula** `3 pt`

Show that the BPS mass formula

$$M^2_{BPS} \;=\; 4\pi v^2\ \ (n_e,\, n_m)\begin{pmatrix} \frac{1}{\operatorname{Im}\tau} & -\frac{\operatorname{Re}\tau}{\operatorname{Im}\tau} \\ -\frac{\operatorname{Re}\tau}{\operatorname{Im}\tau} & \frac{|\tau|^2}{\operatorname{Im}\tau}\end{pmatrix}\begin{pmatrix} n_e \\ n_m\end{pmatrix}\ , \tag{1}$$

is invariant with respect to $SL(2,\mathbb{Z})$ transformations, which act on the charge quanta (n_e, m_e) and the complex coupling constant τ as

$$\begin{pmatrix} n_e \\ n_m\end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} n_e \\ n_m\end{pmatrix}\ , \qquad \tau \mapsto \frac{a\tau + b}{c\tau + d}\ ,$$

respectively.

4.) **Stability of BPS-dyonic states** `3 pt`

We want to show that BPS-dyons (n_e, m_e) are stable, if the charge quanta n_e and m_e are co-prime.

a) Show that the BPS-mass formula (1) defines a norm on the lattice of charge quanta (n_e, m_e). In particular, demonstrate that it fulfills the triangle inequality.

b) Show that the BPS-dyonic state (n_e, m_e) with n_e and m_e co-prime is stable.
Hint: Argue with the triangle inequality of the BPS-norm.