Tutorial Dualities in Field and String Theory Sheet 6

1.) **Moduli space of elliptic curves** `5 pt`
We want to determine the complex structure moduli space of elliptic curves

$$E = \mathbb{C}/\Lambda \ ,$$

given in terms of the rank two lattice $\Lambda \subset \mathbb{C}$ generated by $\omega_1, \omega_2 \in \mathbb{C}$, that is to say the lattice Λ is given by $\Lambda = \{ \, n_1\omega_1 + n_2\omega_2 \in \mathbb{C} \mid (n_1, n_2) \in \mathbb{Z}^2 \}$.

a) Two elliptic curves E and E' have the same complex structure, if there is a holomorphic map $f : E \to E'$. Show that such a map lifts in the covering space $\mathbb{C}$ to an affine map $\tilde{f} : \mathbb{C} \to \mathbb{C}, z \mapsto \alpha\, z + \beta$.
Hint: Use Liouville's theorem, which says that every bounded holomorphic function defined on $\mathbb{C}$ must be constant.

b) Use the result of a) to show that two elliptic curves E and E' associated to two lattices Λ and Λ' have the same complex structure if and only if $\tau = \frac{\omega_1}{\omega_2}$ and $\tau' = \frac{\omega_1'}{\omega_2'}$ are related by a $PSL(2, \mathbb{Z})$ transformation:

$$\tau' = \frac{a\tau + b}{c\tau + d} \qquad \text{with} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

Remark: Choose always lattice generators ω_1 and ω_2 such that $\operatorname{Im} \tau > 0$.

c) Argue that the space of complex structures (called "complex structure moduli space") of elliptic curves are described by the orbits of τ under $PSL(2, \mathbb{Z})$.

d) The $PSL(2, \mathbb{Z})$ orbit space of complex structures is a one-dimensional complex manifold except at two points, where the local structure is that of a disk divided by the action of a finite cyclic group $\mathbb{Z}_k$ of rotations. Such points are called orbifold points. Identify these two points in the fundamental domain $\mathbb{F}$ and determine their cyclic orbifold group $\mathbb{Z}_k$. Which elements of $PSL(2, \mathbb{Z})$ generate these two cyclic groups?

2.) **Adjoint representations of the restricted Lorentz group** `2 pt`
The adjoint representation of the restricted Lorentz group $SO^+(1, 3)$ is generated by antisymmetric tensors $L_{\mu\nu} = -L_{\nu\mu}$. Express the adjoint representation of the Lorentz group in terms of the $\mathfrak{su}_2 \times \mathfrak{su}_2$ spin labels (j, j') of $\mathfrak{sl}_2\mathbb{C}$ as introduced in the lecture.

3.) **Little group of $SL(2, \mathbb{C})$** `3 pt`
Let k_μ be a four-vector with $k^0 > 0,\ k^\mu k_\mu = M^2 \geq 0$. Prove that the little group of k_μ – which is the subgroup of $SL(2, \mathbb{C})$ that leaves k_μ invariant – is isomorphic to

a) $SU(2)$ for $M^2 > 0$ or

b) $\tilde{E}_2$ for $M^2 = 0$.

Here $E_2 \simeq SO(2) \ltimes \mathbb{R}^2$ is the two-dimensional Euclidean group and $\tilde{E}_2 \simeq Spin(2) \ltimes \mathbb{R}^2$ its double cover.
Hint: Choose a convenient k^μ in each case and examine the action of $SL(2, \mathbb{C})$ on the bispinor $\sigma^\mu k_\mu$, where $\sigma^\mu = (1, \vec{\sigma})$ in terms of the Pauli matrices $\vec{\sigma}$.