Tutorial Dualities in Field and String Theory — Sheet 7

1.) **Supersymmetric quantum mechanics** 3 pt

The supersymmetry algebra of quantum mechanics is given by

$$\{Q, Q^\dagger\} = 2H \ , \qquad \{Q, Q\} = 0 \ , \qquad [Q, H] = 0 \ ,$$

where Q is the (fermionic) supercharge and H the Hamiltonian.

a) Show that the quantum mechanical supersymmetry algebra implies a positive energy spectrum.

b) If we diagonalize H by $H\,|n\rangle = E_n\,|n\rangle$, what is the structure of the energy eigenspaces. Distinguish between the possibilities $E_n > 0$ and $E_n = 0$.

c) Using the results of b) to describe the Witten index

$$\mathrm{Tr}\left[(-1)^F\, e^{-\beta H}\right]$$

of supersymmetric quantum mechanical system. Here β is the inverse temperature and F is the fermion number obeying $\{Q, (-1)^F\} = \{Q^\dagger, (-1)^F\} = 0$.
What is the temperature dependence of the Witten index in supersymmetric quantum mechanical systems? What are the implications for a supersymmetric system with a vanishing Witten index?

2.) **Casimir operators of the $N = 1$ super Poincaré algebra** 5 pt

The Casimir operators of the Poincaré algebra are the mass operator $P^2 = P_\mu P^\mu$ and the square $W^2 = W_\mu W^\mu$ of the Pauli-Lubansky operator

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu L^{\rho\sigma} \ ,$$

which is given in terms of the translation and Lorentz generators P_μ and $L_{\mu\nu}$.

a) Show that W^2 is *not* a Casmir operator of the $N = 1$ Super-Poincaré algebra.

b) Show that P^2 and C^2 are the actual Casimir operators of the $N = 1$ Super-Poincaré algebra, where

$$C^2 = C_{\mu\nu} C^{\mu\nu} \ , \qquad C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu \ , \qquad B_\mu = W_\mu - \frac{1}{4}\bar{Q}_{\dot\alpha}\bar\sigma_\mu^{\dot\alpha\beta} Q_\beta$$

Hint: Compute the commutators $[W_\mu, Q_\alpha]$ *and* $[\bar{Q}_{\dot\alpha}\bar\sigma_\mu^{\dot\alpha\beta} Q_\beta, Q_\gamma]$.

3.) **Supersymmetric equations of motion** 2 pt

In the lecture, we considered the on-shell supersymmetry transformations

$$\delta_\xi \phi = \sqrt{2}\xi^\alpha \psi_\alpha \ , \qquad \delta_\xi \psi_\alpha = \sqrt{2}\sigma^\mu_{\alpha\dot\alpha}\bar\xi^{\dot\alpha} P_\mu \phi \ ,$$

of a massless chiral multiplet (ϕ, ψ_α). Use these supersymmetry transformation rules to show that the massless Dirac equations

$$\sigma^\mu_{\alpha\dot\alpha} P_\mu \psi^\alpha = 0 \ ,$$

implies the massless Klein-Gordon equation

$$P_\mu P^\mu \phi = 0 \ .$$