Physikalisches Institut
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Theoretische Physik

Exercise 10
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Exercises on General Relativity and Cosmology

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Due on June 22, 2015

H 10.1 Metric for the charged black hole *(26 points)*

In the lecture you have found the Schwarzschild metric, which is the vacuum solution to Einstein's equations for a spherically symmetric static source. Here, we generalize this discussion to the case in which the source is charged. It is still static and spherically symmetric, hence we make the same ansatz as for the Schwarzschild solution,

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin\theta^2\mathrm{d}\phi^2\right), \tag{1}$$

with $A, B > 0$. As shown in the lecture, the non-zero components of the Ricci tensor then are

$$\begin{aligned} R_{tt} &= \frac{A''}{2B} - \frac{A'B'}{4B^2} - \frac{(A')^2}{4AB} + \frac{A'}{Br}\ , \\ R_{rr} &= -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{(A')^2}{4A^2} + \frac{B'}{Br} \\ R_{\theta\theta} &= -\frac{A'r}{2AB} + \frac{B'r}{2B^2} - \frac{1}{B} + 1\ , \\ R_{\phi\phi} &= \sin\theta^2 R_{\theta\theta}\ . \end{aligned} \tag{2}$$

Due to the presence of charge, we now expect a non-vanishing field strength tensor F. We make the ansatz

$$\begin{aligned} F_{tr} &= -F_{rt} = f(r)\ , \\ F_{\theta\phi} &= -F_{\phi\theta} = g(r)\sin\theta\ , \end{aligned} \tag{3}$$

with all other components vanishing.

a) Calculate the energy momentum tensor associated the field strenght tensor F as defined in eq. (3). Demonstrate that this energy momentum tensor indeed obeys the spherically symmetric static ansatz. *(4 points)*

b) Solve Maxwell's equations (those for curved space-time, and both the homogenous and inhomogenous) to obtain

$$g(r) = c_1\ , \quad f(r) = \sqrt{A(r)\cdot B(r)}\cdot\frac{c_2}{r^2}\ , \tag{4}$$

with constants c_1 and c_2. *(4 points)*

c) Now we move on to solve Einstein's equations in their traced reversed formulation. Show that the energy momentum tensor calculated in item a) is traceless. We thus have to solve (*3 points*)

$$R_{\mu\nu} = \kappa T_{\mu\nu} \ . \tag{5}$$

d) Take a suitable linear combination of the tt and rr component of eq. (5) to derive the relation (*3 points*)

$$A(r) \cdot B(r) = 1 \ . \tag{6}$$

Note: Strictly speaking you will find $A(r) \cdot B(r) =$ constant. As in the lecture, this constant can be rescaled to 1 without loss of generality.

e) Consider the remaining equations to find the differential equations (*4 points*)

$$\begin{aligned} A'' + \frac{2}{r}A' - \kappa f^2 - \kappa\frac{g^2}{r^4} &= 0 \ , \\ rA' + A - 1 + \frac{\kappa}{2r^2}(g^2 + f^2 r^4) &= 0 \ . \end{aligned} \tag{7}$$

f) Solve the differential equations (7) to find the Reissner-Nordström metric

$$\begin{aligned} \mathrm{d}s^2 &= -\Delta \mathrm{d}t^2 + \Delta^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin\theta^2 \mathrm{d}\phi^2\right) , \quad \text{with} \\ \Delta &= 1 - \frac{c_3}{r} + \frac{\kappa}{2} \cdot \frac{c_1^2 + c_2^2}{r^2} \ , \end{aligned} \tag{8}$$

with another constant c_3. (*4 points*)

g) Relate the integration constants c_1, c_2 and c_3 to the mass, electric charge and magnetic charge of the field generating source. To this end, analyse the asymptotics of the Reissner-Nordström metric, the electromagnetic fields stemming from F and recall that the magnetic field of magnetic monopole with magnetic charge M is given by $B(r) = M/r^2$. (*4 points*)

H 10.2 On the Strong Energy Condition (*4 points*)

The *Strong Energy Condition* (SEC) states that the (0,2) energy momentum tensor T has to satisfy the inequality

$$T(X, X) \geq \frac{1}{2} (\mathrm{tr}_g T) \, g(X, X) \tag{9}$$

for every time-like vector field X. Show that the energy momentum tensor for vacuum energy,

$$T_{\text{vac}} = -\rho_{\text{vac}} \cdot g \quad \text{with } \rho_{\text{vac}} > 0 \ , \tag{10}$$

violates the SEC.