Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 11
June 22, 2015
Summer term 2015

Exercises on General Relativity and Cosmology

Dr. Hans Jockers, Prof. Dr. Hans-Peter Nilles

http://www.th.physik.uni-bonn.de/klemm/gr_15/

–Comments–

On this sheet you can achieve 36 points. Six of these points, those marked with a star, are extra points that do not count for the 50% threshold.

–Home Exercises–

Due on June 29, 2015

H 11.1 Killing vector fields and geodesics on S^2 *(12+*3 points)*

As in the lecture, a Killing vector field K is defined as a vector field on a (pseudo) riemannian manifold (M, g) that obeys

$$g\left(\nabla_X K, Y\right) + g\left(X, \nabla_Y K\right) = 0 \tag{1}$$

for any vector fields X and Y. Here, we consider the Killing vector fields on the two sphere $M = S^2$. According to the lecture these are

$$\begin{aligned} K_1 &= -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi \ , \\ K_2 &= +\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi \ , \\ K_3 &= \partial_\phi \ , \end{aligned} \tag{2}$$

Here, θ and ϕ are local coordinates of S^2, in terms of which the metric reads

$$g = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 \ . \tag{3}$$

As we will see below, Killing vector fields can be used to find the geodesics on M.

a) Consider a geodesic $\mathcal{C}$ parameterized by λ, where λ is affinely related to the arc length. Deduce from the defintion (1) that for a Killing vector field K

$$g\left(K, \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\lambda}\right) = \text{constant along the geodesic } \mathcal{C} \ . \tag{4}$$

These are first order differential equations for the geodesic. *(2 points)*

b) On sheet 5 we have introduced the Lie bracket (commutator) of vector fields, denoted as $[X, Y]$. Show that

$$[K_i, K_j] = -\epsilon_{ijk} K_k \ , \quad i, j, k = 1, 2, 3 \ , \tag{5}$$

with the totally antisymmetric ϵ-tensor that is normalized by $\epsilon_{123} = 1$. *(3 points)*

c) Note that the vector fields $-iK_j$ for $j = 1, 2, 3$ and $i^2 = -1$ fulfill the angular momentum algebra. Can you think of a reason for this? *(*3 points)*

d) For the Killing vector fields given in eq. (2), write eqs. (4) in terms of the local coordinates. Label the constant appearing on the right hand side of (4) for $K = K_i$ as L_i *(3 points)*

e) Combine the equations found in the previous item to arrive at an equation, in which θ, ϕ and the L_i but no derivatives of the geodesic appear. This equation can (at least locally) be solved to yield the geodesics in the form $\theta(\phi)$ or $\phi(\theta)$. *(2 points)*

f) Find the geodesics for the three cases in which only one L_i is non-zero. *(2 points)*

H 11.2 Relativistic and gravitational redshift *(13 points)*

One of the striking predicitions of general relativity is the gravitational redshift of light, which has been experimentally verified by the *Pound-Rebka* experiment. In this exercise we elaborate on what has been said about this experiment in the lecture.

As usual, we set $c = \hbar = 1$. Consider an observer O with four-velocity U and a photon moving along a geodesic $\mathcal{C}$, which is parameterized such that its tangent vector $\mathrm{d}\mathcal{C}/\mathrm{d}\lambda$ equals the photon's four-momentum P. It has been argued in the lecture that the observer O measures the photon's angular frequency as

$$\omega_O = -g\left(U, \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\lambda}\right) , \tag{6}$$

where g is the space-time metric at the point of measurement.

a) Let us first consider the relativistic Doppler effect, which is already present in special relativity, i.e. we take $g = \eta$ with the Minkowski metric η. Write down the four-velocity U_1 of an observer O_1 in its rest-frame. In this coordinate system, further write down the four-momentum P of a photon that travels along the unit direction $\vec{e}_\gamma$ and whose angular frequency is by O_1 measured to be ω_1. *(3 points)*

b) In the same coordinate system, write down the four-velocity U_2 of a second observer O_2, who by O_1 is measured to travel with velocity $\vec{v} = v\vec{e}_2$, $|\vec{v}| = v$. Show that O_2 measures the photon's angular frequency as *(3 points)*

$$\omega_2 = \gamma(v)\omega_1 \left(1 - v\vec{e}_\gamma \cdot \vec{e}_2\right) . \tag{7}$$

c) Now we consider to the gravitational redshift in the Schwarzschild metric. Look up this metric in your lecture notes and write down the four-veloctiy of an observer that is static in the Schwarzschild coordinates and sits at radius r. Consider two of these observers, one at $r = r_1$, the other one at $r = r_2$. Denote the frequencies they measure as $\omega(r_1)$ and $\omega(r_2)$ respectively and find an expression for the ratio $\omega(r_1)/\omega(r_2)$. *(2 points)*

Hint: Recall from the lecture that for radial photons the relation

$$\left(1 - \frac{R_s}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\lambda} = E \tag{8}$$

holds. Here, λ is the same as the one in and above eq. (6)*. Further, E is a constant, namely the energy (frequency) that an observer resting at infinity would assign to the photon.*

d) We are interested in the following situation: The Schwarzschild metric is used as an approximation for the metric outside the surface of earth. Further, we take $r_2 = R_0$ as the radius of Earth and set $r_1 = r_2 + h$ with the height h of a tall building. Use this to appropriately approximate your result from item c). You should find

$$\omega(r_1)/\omega(r_2) = 1 - g \cdot h \ , \tag{9}$$

where g is the gravitational acceleration on the surface of Earth. *(2 points)*

e) In the Pound-Rebka experiment, a photon emitting ^{57}Fe source was mounted on a loudspeaker at the top of a tower. Both the photon emission and the movement of the source due to turning on the loudspeaker is along the tower (perpendicular to the surface of Earth). At the bottom of the tower, a ^{57}Fe source is mounted above a detector. By looking for the minimum of the counting rate, the velocity of the source at which the relativistic Doppler effect cancels the gravitational redshift has been measured. Find the velocity (both the absolute value as well as the direction) at which this happens to first order in v and h. *(3 points)*

H 11.3 More on energy conditions *(5+*3 points)*

On sheet 10 we have consider the Strong Energy condition. In the lecture, two other energy conditions have been mentioned,

$$\begin{aligned} \text{Null Energy condition (NEC):} \quad & T(X,X) \geq 0 \quad \text{for all light-like vectorfields } X \ , \\ \text{Weak Energy condition (WEC):} \quad & T(X,X) \geq 0 \quad \text{for all time-like vectorfields } X \ . \end{aligned} \tag{10}$$

In this exercise, we want to apply these conditions to the energy momentum tensor of a perfect fluid, which in components reads

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{11}$$

with the fluid's four-velocity U, as well as the energy- and pressure-densities ρ and p.

a) Show that the NEC for T as in eq. (11) implies $\rho + p \geq 0$. *(2 points)*

b) Proof that the WEC implies the NEC directly from the definitions in eq. (10), i.e. without any other assumption on T. *(*3 points)*

c) Show that the WEC for T as in eq. (11) implies $\rho + p \geq 0$ and $\rho \geq 0$. You may use item b) for this. *(3 points)*