Exercises on General Relativity and Cosmology

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Due on July 6, 2015

H 12.1 Falling freely inside a black hole *(15 points)*

Consider a particle that has crossed the event horizon of a Schwarzschild black hole at $r = R_S$ and is then falling freely within the event horizon. Work with the standard Schwarzschild coordinates (t, r, θ, ϕ) and denote the particle's proper time by τ.

a) Show that

$$\left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| \geq \sqrt{\frac{R_S}{r} - 1} \ . \tag{1}$$

When is this bound saturated? *(7 points)*

b) Calculate the maximum lifetime of a particle falling freely from $r = R_S$ to $r = 0$. Plug in numbers to express this in seconds for a black hole with one solar mass, $M = 2 \cdot 10^{30}$ kg. *(4 points)*

c) Can an observer that also travels from $r = R_S$ to $r = 0$ but not on a geodesic have a longer lifetime than the one calculated in item b)? Justify your answer. *(4 points)*

H 12.2 Particle motion in the presence of electromagnetic fields *(15 points)*

As you have shown in H 9.1 b), in the presence of an electromagnetic field a particle of charge q and mass m moves along a curve $\mathcal{C}$ that obeys

$$\nabla_{\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\tau}} \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\tau} = \frac{q}{m} F^\mu{}_\nu(\mathcal{C}(\tau)) \frac{\mathrm{d}\mathcal{C}^\nu}{\mathrm{d}\tau} \partial_\mu \ , \tag{2}$$

in terms of the particle's proper time τ.

a) Given a Killing vector field K, show that *(5 points)*

$$\nabla_{\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\tau}} g\left(\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\tau}, K\right) = g\left(\frac{q}{m} F^\mu{}_\nu(\mathcal{C}(\tau)) \frac{\mathrm{d}\mathcal{C}^\nu}{\mathrm{d}\tau} \partial_\mu, K\right) \ . \tag{3}$$

Now imagine that such a particle is moving in the field of a Reissner-Norström black hole of electric charge Q, magnetic charge P and mass M. From exercise H 10.1 recall the Reissner-Norström metric,

$$\mathrm{d}s^2 = -\Delta \mathrm{d}t^2 + \Delta^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \ , \ \Delta \ = 1 - \frac{2GM}{r} + \frac{G\left(Q^2 + P^2\right)}{r^2} \ , \tag{4}$$

and the non-vanishing components of the field strength tensor,

$$F_{tr} = -F_{rt} = -\frac{Q}{r^2} \ , \quad F_{\theta\phi} = -F_{\phi\theta} = P \sin\theta \ . \tag{5}$$

b) Use the Killing vector field corresponding to time-translation symmetry and eq. (3) to show that the energy

$$E = m\left(1 - \frac{2GM}{r} + \frac{G\,(Q^2 + P^2)}{r^2}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{q\,Q}{r} \tag{6}$$

is conserved. *(5 points)*

c) Use the Killing vector field corresponding to the symmetry under rotations along the angle ϕ and eq. (3) to show that the angular momentum

$$L_\phi = m\,r^2 \sin^2\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}\tau} - qP\cos\theta \tag{7}$$

is conserved. *(5 points)*