Exercises on General Relativity and Cosmology

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–Comments–

All points that you can get on this sheet are additional in the sense that they do not increase the 50% threshold of maximally achievable points. Nevertheless, the content of this sheet is still relevant for the exam.

–Home Exercises–

Due on July 13, 2015

H 13.1 Robertson-Walker universe and the Friedmann equations *(20 points)*

In order to describe the evolution of the universe as a whole, we consider the Robertson-Walker metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \right) \right] . \tag{1}$$

Further, we model the matter and energy content of the universe by a perfect fluid that is comoving with respect to the coordinate system (t, r, θ, ϕ), hence

$$T = [\rho(t) + p(t)]\mathrm{d}t^2 + p(t) \cdot g . \tag{2}$$

a) Show that the non-vanishing Christoffel symbols of the metric (1) are

$$\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1-\kappa r^2}, & \Gamma^1_{11} &= \frac{\kappa r}{1-\kappa r^2}, \\
\Gamma^0_{22} &= a\dot{a}r^2, & \Gamma^0_{33} &= a\dot{a}r^2 \sin^2\theta, \\
\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} &= \frac{\dot{a}}{a}, & \Gamma^1_{33} &= \sin^2\theta \, \Gamma^1_{22} = -r(1-\kappa r^2)\sin^2\theta , \\
\Gamma^2_{12} = \Gamma^3_{13} &= \frac{1}{r}, & \Gamma^2_{33} &= -\sin^2\theta \, \Gamma^3_{23} = -\sin\theta \, \cos\theta ,
\end{aligned} \tag{3}$$

and those related to these by symmetry in the lower indices. (*5 points*)

b) Show that energy-momentum conservation, most conveniently calculated in the form $\nabla_\mu T^\mu{}_\nu = 0$, is equivalent to

$$\partial_0 \rho = -3\frac{\dot{a}}{a}(\rho + p) . \tag{4}$$

Bring this to the form

$$\partial_0(\rho \cdot a^3) = -p \, \partial_0 a^3 , \tag{5}$$

and interpret this in terms of an equation familiar from thermodynamics. (*4 points*)

c) Show that the off diagonal elements of the Ricci tensor are zero. For most of them this easily follows from a symmetry argument. Further show (*5 points*)

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1 - \kappa r^2}, \quad R_{22} = \frac{R_{33}}{\sin^2\theta} = r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa) \ . \tag{6}$$

d) Show that the Ricci scalar is (*2 points*)

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right] \ . \tag{7}$$

e) Now consider Einstein's equation in its trace-reversed form. Show that its $\mu\nu = 00$ component gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \ . \tag{8}$$

Further show that the $\mu\nu = ij$ components give, when combined with eq. (8),

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \ . \tag{9}$$

Why is there only one distinct equation from $\mu\nu = ij$? Together eqs. (8) and (9) are known as the Friedmann equations. (*4 points*)

H 13.2 A Robertson-Walker universe dominated by radiation (*10 points*)

In this exercise we consider the evolution of the scale factor for a Robertson-Walker universe dominated by radiation. Denote the present values of a, $\dot{a}/a$ and ρ by a_0, H_0 and ρ_0. Further, the deceleration parameter $q = -\ddot{a}a/\dot{a}^2$ has the present day value q_0. Domination by radiation means that we assume the energy and matter content of the universe to consist of photons (and/or ultra-relativistic matter) only. Recall that the equation of state for photons and ultra-relativistic matter reads

$$p = \frac{\rho}{3} \ . \tag{10}$$

Since we consider a Robertson-Walker universe, the equations derived in exercise H 13.1 are valid in this exercise too.

a) Show that (*1 point*)

$$q_0 = \frac{8\pi}{3}\frac{G\rho_0}{H_0^2} \ . \tag{11}$$

b) Show that (*2 points*)

$$\rho = \rho_0 \cdot \left(\frac{a_0}{a}\right)^4 \ . \tag{12}$$

c) Show that (*3 points*)

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2\left[1 - q_0 + q_0\left(\frac{a_0}{a}\right)^2\right] \tag{13}$$

d) Assume $\dot{a} \geq 0$ (we also know $a_0 > 0$ and $q_0 \geq 0$) and show that *(2 points)*

$$t = \frac{1}{H_0} \int_0^{a/a_0} \frac{x \, \mathrm{d}x}{\sqrt{(1 - q_0)x^2 + q_0}} \,. \tag{14}$$

Give a formula for the age of the universe.

e) Show that

$$t = \frac{1}{(1 - q_0)H_0} \left[\sqrt{(1 - q_0) \left(\frac{a}{a_0} \right)^2 + q_0} - \sqrt{q_0} \right] \,. \tag{15}$$

Convince yourself that this expression makes sense for $q_0 = 1$ too and show how the equation simplifies in this case. What is the age of the universe? Give this for general q_0 as well as the simplified version for $q_0 = 1$? Further show *(2 points)*

$$\left(\frac{a}{a_0} \right)^2 = H_0 t \left[2\sqrt{q_0} - H_0 t (1 - q_0) \right] \,. \tag{16}$$