Exercises on General Relativity and Cosmology

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Due on April 20, 2015

H 2.1 The Lorentz group *(15 points)*

We consider four-dimensional Minkowski space $\mathbb{R}^{1,3}$, which is $\mathbb{R}^4$ equipped with the Minkowski metric η. This is a symmetric, non-degenerate bilinear form $\eta : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ defined by

$$\eta(e_\mu, e_\nu) \equiv \eta_{\mu\nu} = \begin{cases} -1 & \text{for } \mu = \nu = 0 \\ +1 & \text{for } \mu = \nu = 1, 2, 3 \end{cases} \tag{1}$$

for the standard orthonormal basis $\{e_0, e_1, e_2, e_3\}$ on $\mathbb{R}^4$. Using linearity we then find

$$\eta(x, y) = x^{\mathrm{t}} \cdot \tilde{\eta} \cdot y \quad \text{for } x, y \in \mathbb{R}^{1,3}, \tag{2}$$

where $\tilde{\eta}$ is a matrix with entries $\eta_{\mu\nu}$. From now, we identify $\tilde{\eta}$ and η with each other and do not distinguish between them.
For $x, y \in \mathbb{R}^{1,3}$ we write $x \cdot y = \eta(x, y)$ and $x^2 = x \cdot x$. The postulates of special relativity imply that transformations Λ relating two inertial frames, so called Lorentz transformations, preserve the spacetime distance, i.e.

$$(x - y)^2 = (\Lambda(x - y))^2 \quad \text{for all} \quad x, y \in \mathbb{R}^{1,3}\,. \tag{3}$$

This leads to the definition of the *Lorentz group*

$$\mathrm{O}(1, 3) = \{\Lambda \in \mathrm{GL}(4, \mathbb{R}) \,|\, \Lambda^{\mathrm{t}} \eta \Lambda = \eta\}\,. \tag{4}$$

a) Show that $\Lambda \in \mathrm{O}(1, 3)$ indeed fulfills eq. (3). (*1 point*)

b) Show that $\mathrm{O}(1, 3)$ indeed is a group. (*3 points*)

c) Show that $\Lambda^{\mathrm{t}} \eta \Lambda = \eta$ written in components reads $\eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}$. (*1 point*)

d) Embed the group of three-dimensional rotations into $\mathrm{O}(1, 3)$. (*1 point*)

e) Show that $|\Lambda^0{}_0| \geq 1$ and that $|\det \Lambda| = 1$. With this argue that the Lorentz group consists of four branches (which are not continuously connected to each other). Hint: Use $\det(\mathbb{1} + \epsilon\lambda) = 1 + \epsilon \operatorname{tr} \lambda + \mathcal{O}(\epsilon^2)$. (*3 points*)

f) Show that the subset $\mathrm{SO}^+(1, 3) = \{\Lambda \in \mathrm{O}(1, 3) \,|\, \det \Lambda = 1\,,\, \Lambda^0{}_0 \geq 1\}$ forms a subgroup of $\mathrm{O}(1, 3)$, called the *proper orthochronous Lorentz group.* (*2 points*)

g) Identify the Lorentz transformations for time and parity reversal and relate them to the respective branches. (*1 point*)

Consider two inertial frames, K and K'. When K' moves in K with velocity v in positive x_1 direction, the Lorentz transformation from K to K' is ($c = 1$)

$$\Lambda_{x_1}(v) = \begin{pmatrix} \gamma & -\gamma \cdot v & 0 & 0 \\ -\gamma \cdot v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

Transformation of this type are called *boosts*. We introduce the *rapidity* ϕ by $v = \tanh \phi$.

h) Rewrite $\Lambda_{x_1}(v)$ from eq. (5) in terms of the rapidity. (*1 point*)

i) Consider two succsessive boost, both in the x_1 direction but with different velocities. Find the rapidity of the composite boost. Deduce the relativistic rule for addition of velocities. (*2 points*)

H 2.2 Classical electrodynamics *(15 points)*

In this exercise we consider the field theoretical formulation of classical electrodynamics. The electromagnetic field is described in terms of a vector field

$$A : \mathbb{R}^{1,3} \longrightarrow \mathbb{R}^{1,3}, \tag{6}$$

with components A^μ. These are related to the components of the field strength tensor F via

$$F_{\mu\nu} = \frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu. \tag{7}$$

The particles of mass m_i and charge q_i are described by their trajectories,

$$x_i : I_k \subset \mathbb{R} \longrightarrow \mathbb{R}^{1,3} \quad \text{for } i = 1, ..., N = \text{number of particles} \tag{8}$$

$$\sigma_i \longmapsto x_i(\sigma_i), \tag{9}$$

which are parameterised by an arbitrary curve parameter σ_i and whose components are x_i^μ. The action is

$$S[x_i, A] = -\sum_{i=1}^{N} \int_{I_k} \mathrm{d}\sigma_i \left(m_i \sqrt{-\eta_{\alpha\beta} \dot{x}_i{}^\alpha(\sigma_i) \dot{x}_i{}^\beta(\sigma_i)} - q_i\, A_\alpha(x_i(\sigma_i)) \dot{x}_i{}^\alpha \right) \tag{10}$$

$$- \frac{1}{4} \int_{\mathbb{R}^{1,3}} \mathrm{d}^4 x\, F_{\alpha\beta}(x) F^{\alpha\beta}(x), \tag{11}$$

where $\dot{x}_i = \frac{\mathrm{d}}{\mathrm{d}\sigma_i} x_i$ and integration on Minkowski space is the same as on $\mathbb{R}^4$.

a) Shortly comment on the significance of each term in S. (*1 point*)

b) Take the variation of S with respect to x_i^μ in order to derive the Einstein-Lorentz equation (*2 points*)

$$m_i \frac{\mathrm{d}}{\mathrm{d}\sigma_i} \frac{\dot{x}_i{}^\mu(\sigma_i)}{\sqrt{-\eta_{\alpha\beta} \dot{x}_i{}^\alpha(\sigma_i) \dot{x}_i{}^\beta(\sigma_i)}} = q_i\, F^\mu{}_\nu(x_i(\sigma_i)) \dot{x}_i{}^\nu. \tag{12}$$

c) Rewrite the second term in S, the term where q_i appears, in terms of the charge-current density (*1 point*)

$$j^\mu(x) = \sum_{i=1}^{N} q_i \int \mathrm{d}\sigma_i \, \delta^{(4)}(x - x_i(\sigma_i)) \, \dot{x}_i{}^\mu(\sigma_i). \tag{13}$$

d) Take the variation of S with respect to A_μ to derive the inhomogenous Maxwell's equations (*2 points*)

$$\frac{\partial}{\partial x^\mu} F^{\nu\mu}(x) = j^\nu(x). \tag{14}$$

e) Use the definition of the field strength tensor, eq. (7), to show the homogenous Maxwell's equations (*2 points*)

$$\frac{\partial}{\partial x^\alpha} F_{\mu\nu} + \frac{\partial}{\partial x^\mu} F_{\nu\alpha} + \frac{\partial}{\partial x^\nu} F_{\alpha\mu} = 0. \tag{15}$$

Now take $A^\mu = (\phi, \vec{A})$ with ϕ and $\vec{A}$ such that $\vec{B} = \mathrm{rot}\,\vec{A}$ and $\vec{E} = -\,\mathrm{grad}\,\phi - \dot{\vec{A}}$. Further, $j^\mu = (\rho, \vec{j})$ with the charge-density ρ and current-density $\vec{j}$.

f) Express the components of the field strength tensor, $F_{\alpha\beta}$, in terms of the components of the electric and magnetic field, $\vec{E}$ and $\vec{B}$. (*1 point*)

g) Show that eq. (14) indeed gives the inhomogenous Maxwell's equations: (*2 points*)

$$\mathrm{div}\,\vec{E} = \rho, \qquad \mathrm{rot}\,\vec{B} - \dot{\vec{E}} = \vec{j}. \tag{16}$$

h) Show that eq. (15) indeed gives the homogenous Maxwell's equations: (*2 points*)

$$\mathrm{div}\,\vec{B} = 0, \qquad \mathrm{rot}\,\vec{E} + \dot{\vec{B}} = 0. \tag{17}$$

i) Parameterise eq. (12) by time, i.e. $\sigma = x^0 = t$, and show that it reduces to (*2 points*)

$$m \frac{\mathrm{d}}{\mathrm{d}t} \frac{\vec{v}}{\sqrt{1 - v^2}} = q\left(\vec{E} + \vec{v} \times \vec{B}\right). \tag{18}$$