Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 3
April 20, 2015
Summer term 2015

Exercises on General Relativity and Cosmology

Dr. Hans Jockers, Prof. Dr. Hans-Peter Nilles

http://www.th.physik.uni-bonn.de/klemm/gr_15/

–Home Exercises–

Due on April 27, 2015

H 3.1 Temporal order of events in relativity *(6 points)*

An observer sees three events in the order ABC, whereas a second observer sees them in order CBA.

a) In two-dimensional (1+1) Minkowski space, can there be a third observer for whom the events appear in order ACB? Argue for your answer, for example by drawing a spacetime diagram. (*4 points*)

b) Does this carry over to higher-dimensional Minkowski space? (*2 points*)

H 3.2 Linear algebra, indices and tensors *(15 points)*

In this exercise we want to review some basics of linear algebra, familiarize ourselves with the meaning of upper and lower indices and with the notion of tensors. The applications to relativity are subject of H 3.3.

Consider a real vector space V of dimension $n < \infty$. For any vectorspace we define its dual vector space by $V^* = \{w : V \to \mathbb{R} \,|\, w \text{ linear}\}$. Upon introduction of a basis on V, $\mathbb{B}_1 = \{e_i \in V \,|\, \mathrm{i} = 1, ..., n\}$, any vector $v \in V$ can be expanded in this basis. This means that there are unique numbers v^i, $i = 1, \ldots, n$ for which

$$v = \sum_{i=1}^{n} v^i e_i \equiv v^i e_i \tag{1}$$

holds. The numbers v^i are referred to as the components of v in the basis $\mathbb{B}_1$ and one sometimes identifies the list of numbers v^i with the vector v. This, however, works only as long as a specific basis is fixed, whereas the vector v itself exists independently of any basis. Once a basis of V has been chosen, there is a natural choice of basis of V^*. This is the dual basis $\mathbb{B}_1^* = \{e^i \in V^* \,|\, \mathrm{i} = 1, ..., n\}$ defined by

$$e^i\,(e_j) = \delta^i_j \quad \forall i, j \in \{1, ..., n\}. \tag{2}$$

Now, any $w \in V^*$ can be written as

$$w = \sum_{i=1}^{n} w_i e^i \equiv w_i e^i, \tag{3}$$

in terms of unique numbers w_i, $i = 1, \ldots, n$.

a) Show that a dual vector is uniquely specified by its values on a basis of V, hence eq. (2) indeed specifies a set of dual vectors. Further show that $\mathbb{B}_1^*$ is indeed a basis of V^*, from which we deduce $V \simeq V^*$. *(2 points)*

b) Let $\mathbb{B}_2 = \{\tilde{e}_i \in V \,|\, \mathrm{i} = 1, ..., n\}$ be a second basis of V and $\mathbb{B}_2^* = \{\tilde{e}^i \in V^* \,|\, \mathrm{i} = 1, ..., n\}$ the associated dual basis. Write $e_i = (e_i)^j \tilde{e}_j$ and $e^i = (e^i)_j \tilde{e}^j$. Relate the components of $v \in V$ in $\mathbb{B}_1$ (and of $w \in V^*$ in $\mathbb{B}_1^*$) to those in $\mathbb{B}_2$ (and $\mathbb{B}_2^*$). *(1 point)*

c) With the same v and w, calculate $w(v)$ in both bases. Does the result depend on the basis? Deduce *(1 point)*

$$(e^i)_k (e_j)^k = \delta^i_j. \tag{4}$$

d) Let us now consider the bidual space $(V^*)^*$, which is the dual space of the dual space. Show that

$$\begin{aligned} \alpha :\, V &\longrightarrow (V^*)^* \\ v &\longmapsto \alpha(v) \text{ defined by } (\alpha(v))(w) = w(v) \text{ for all } w \in V^* \end{aligned} \tag{5}$$

is an isomorphism of vectorspaces. For surjectivity use that for linear maps the formula $\dim V = \dim(\ker \alpha) + \dim(\operatorname{im} \alpha)$ holds. *(2 points)*

Since α does not make reference to any basis, it is called a canonical isomorphism. Thus V and $(V^*)^*$ are regarded as the same space and are not distinguished.
Although we know $V \simeq V^*$ as well, there is in general no preferred choice of isomorphism between V and V^*. The situation changes if V is equipped with a symmetric non-degenerate[1] bilinear form $\beta : V \times V \longrightarrow \mathbb{R}$. Then it is natural to define the isomorphisms

$$\begin{aligned} \phi_1 :\, V &\longrightarrow V^* \\ v &\longmapsto \phi_1(v) = \beta(v, \cdot), \text{ declared by } (\phi_1(v))(w) = \beta(v, w) \text{ for all } w \in V, \\ \phi_2 :\, V^* &\longrightarrow V \\ w &\longmapsto \phi_2(w) \text{ defined by } \beta(\phi_2(w), v) = w(v) \text{ for all } v \in V. \end{aligned} \tag{6}$$

Write $\beta_{ij} = \beta(e_i, e_j)$ and define the numbers β^{ij} by $\beta^{ij}\beta_{jk} = \delta^i_j$.

e) Show that the components of $\phi_1(v)$ in $\mathbb{B}_1^*$ are related those in $\mathbb{B}_1$ by *(1 point)*

$$\phi_1(v)_i = \beta_{ij} v^j. \tag{7}$$

f) Show $\phi_2 \circ \phi_1 = \mathrm{id}_V$ and $\phi_1 \circ \phi_2 = \mathrm{id}_{V^*}$. Deduce that the components of $\phi_2(w)$ in $\mathbb{B}_1$ are related to those of w in $\mathbb{B}_1^*$ by *(2 points)*

$$\phi_2(w)^i = \beta^{ij} w_j. \tag{8}$$

This allows for changing back and forth between V and V^*. Application of ϕ_1 is called lowering an index and application of ϕ_2 raising an index.

[1]This is to ensure that ϕ_1 and ϕ_2 are isomorphisms.

So far we have encountered two objects, vectors (referred to as contravariant) and dual vectors (referred to as covariant). Let us now look at a generalization: A (k,l)-tensor T (k times contra- and l times covariant) is a multilinear map

$$T : \underbrace{V^* \times \ldots \times V^*}_{k \text{ times}} \times \underbrace{V \times \ldots \times V}_{l \text{ times}} \longrightarrow \mathbb{R}, \tag{9}$$

and the space of (k,l)-tensors is denoted $T^{k,l}$. The components of T with respect to the bases $\mathbb{B}_1$ and $\mathbb{B}_1^*$ are

$$T^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \equiv T(e^{i_1}, \ldots, e^{i_k}, e_{j_1}, \ldots, e_{j_l}). \tag{10}$$

The order of indices is significant, because T may answer differently on different arguments. Upper (lower) indices can be lowered (raised) with ϕ_1 (ϕ_2).

g) What type of tensors are scalars, dual vectors and β? What type of tensors are vectors and why is that so? *(1 point)*

h) Why is a tensor uniquely specified by its components? *(1 point)*

i) Find a basis of $T^{k,l}$ in terms of the basis vectors in $\mathbb{B}_1$ and $\mathbb{B}_1^*$. What is the dimension of $T^{k,l}$? *(2 points)*

j) Relate the components of T in $\mathbb{B}_1$ and $\mathbb{B}_1^*$ to those in $\mathbb{B}_2$ and $\mathbb{B}_2^*$. *(2 points)*

H 3.3 Application to relativity *(9 points)*

In special relativity spacetime $M = \mathbb{R}^{1,3}$ is equipped with the Minkowski metric η.

a) What object in H 3.2 is η associated to? *(0.5 points)*

Since η is of Lorentzian signature, i.e. it has one negative and three positive eigenvalues, the indices are denoted by Greek and not Latin letters (which are reserved for Euclidean signature).

b) Why can spacetime not play the role of V in general relativity? *(1 point)*

Instead, the tangent space T_pM at the point $p \in M$ plays the role of V (pointwise). Given coordinates on a patch $U \subset M$ around p, $x^\mu : U \longrightarrow \mathbb{R}^4$ with $\mu = 1, \ldots, 4$, the tangent space is spanned by the partial derivates with respect to the coordinates, i.e. $\partial_\mu = \frac{\partial}{\partial x^\mu}$. The duals of ∂_μ are denoted by $\mathrm{d}x^\mu$, they are elements of the cotangent space T_p^*M.

c) Does μ in x^μ label a vector or dual vector component, or a set of maps? *(0.5 points)*

d) What object in H 3.2 is ∂_μ associated to? *(1 point)*

In special relativity it is convenient to work with inertial frames. Consider two inertial frames — frame A with coordinates x^μ and frame B with y^μ — that are related by a Lorentz transformation $y^\mu = \Lambda^\mu{}_\nu x^\nu$.

e) Express $\partial'_\mu = \frac{\partial}{\partial y^\mu}$ in terms of the ∂_μ. *(2 points)*

f) In item b) of H 3.2 we have looked at a change of basis. What is the analog to this here: The change from x^μ to y^μ or from ∂_μ to ∂'_μ? *(1 point)*

g) Express $\mathrm{d}y^\mu$ (the dual of ∂'_μ) in terms of the $\mathrm{d}x^\mu$. *(1 point)*

h) Let the components of a (k,l)-tensor T in frame A be $T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l}$. Show that the components in frame B are *(2 points)*

$$T^{\mu'_1\dots\mu'_k}{}_{\nu'_1\dots\nu'_l} = \Lambda^{\mu'_1}{}_{\mu_1}\dots\Lambda^{\mu'_k}{}_{\mu_k}(\Lambda^{-1})^{\nu_1}{}_{\nu'_1}\dots(\Lambda^{-1})^{\nu_l}{}_{\nu'_l}T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l}. \tag{11}$$

We have stated above, that T_pM plays the role of V. This means that V now depends on the point in spacetime, thus it is natural to consider tensor fields,

$$\mathcal{T}^{k,l} : M \longrightarrow T^{k,l}. \tag{12}$$

A tensor field of (k,l) type assigns a (k,l)-tensor to each point in spacetime.