Physikalisches Institut
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Theoretische Physik

Exercise 5
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Exercises on General Relativity and Cosmology

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–Home Exercises–

Due on May 11, 2015

H 5.1 The Lie bracket of vector fields *(12 points)*

A smooth vector field X on a manifold M fulfills the two conditions

$$\begin{array}{llll} \text{Linearity:} & X(\alpha f + \beta g) = \alpha X(f) + \beta X(g) & \text{with} & \alpha, \beta \in \mathbb{R},\ f, g \in C^\infty(M) \\ \text{Leibniz rule:} & X(f \cdot g) = f \cdot X(g) + g \cdot X(f) & \text{with} & f, g \in C^\infty(M)\ . \end{array} \tag{1}$$

In general, maps with the properties (1) are called *derivations*. Given two vector fields X and Y we define a new vector field $[X, Y]$, the *Lie bracket* or *commutator* of X and Y, by

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \quad \text{for} \quad f \in C^\infty(M). \tag{2}$$

a) Show in two ways that $[X, Y]$ is indeed a vector field:

   i) Prove that $[X, Y]$ is a derivation. *(3 points)*

   ii) Write $[X, Y]$ in terms of components and show that they transform as those of a vector field unter change of coordinates. *(2 points)*

   Note that neither XY nor YX is a vector field.

b) Show that the Lie bracket is

   i) skew-symmetric, $[X, Y] = -[Y, X]$, and *(1 point)*

   ii) satisfies the Jacobi identity, $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$. *(2 points)*

c) Consider $\mathbb{R}^2$ equipped with some coordinates x^1, x^2. Calculate the Lie bracket of the coordinate vector fields $\partial_1 = \frac{\partial}{\partial x^1}$ and $\partial_2 = \frac{\partial}{\partial x^2}$. *(1 point)*

d) Find an example of two nowhere-vanishing, (at each point) linearly independent vector fields in $\mathbb{R}^2$ whose Lie bracket does not vanish. Note that these two vector fields provide a basis for the tangent space at each point. Due to your findings in item c) they can, however, not be coordinate vector fields. *(3 points)*

Item c) shows that a set of vector fields can only be a set of coordinate vector fields if their mutual Lie brackets vanish. In fact, the reverse is also true: Given n vector fields $X_1, \ldots, X_n$ defined on an open set U in an n-dimensional manifold M that are linearly independent for all $p \in U$ and whose mutual Lie brackets vanish, $[X_k, X_l] = 0$ for all $k, l = 1, \ldots n$, then there exists a local coordinate system $x^1, \ldots, x^n$ such that $X_k = \partial_k$.

H 5.2 Explicit calculations of Christoffel symbols *(18 points)*

Consider a manifold M with metric tensor g. The *Christoffel symbols* are defined by

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \,, \tag{3}$$

where $g_{\mu\nu}$ are the components of g in some coordinate system and $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$.

a) On sheet 4 we considered the two-sphere S^2 and the torus T^2 embedded in $\mathbb{R}^3$, as well as de Sitter space embedded in $\mathbb{R}^{1,4}$. By pulling back the metric on the ambient space with the respective inclusion maps, we obtained the induced metrics

$$\begin{aligned} \mathrm{d}s^2_{S^2} &= R^2 \Big(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \Big) \,, \\ \mathrm{d}s^2_{T^2} &= r^2 \mathrm{d}\theta^2 + \Big(R + r\cos\theta \Big)^2 \mathrm{d}\phi^2 \,, \\ \mathrm{d}s^2_{dS^4} &= -\mathrm{d}t^2 + \alpha^2 \cosh^2(t/\alpha) \Big[\mathrm{d}\chi^2 + \sin^2\chi \Big(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \Big) \Big] \end{aligned} \tag{4}$$

Calculate the Christoffel symbols for these metrics with eq. (3). *(2+2+4 points)*

b) Consider a *diagonal* metric, i.e., $g_{\mu\nu} = 0$ for $\mu \neq \nu$. Show that in this case the Christoffel symbols are given by

$$\begin{aligned} \Gamma^{\lambda}{}_{\mu\nu} &= 0 \,, \\ \Gamma^{\lambda}{}_{\mu\mu} &= -\frac{1}{2} (g_{\lambda\lambda})^{-1} \partial_\lambda g_{\mu\mu} \,, \\ \Gamma^{\lambda}{}_{\mu\lambda} &= \partial_\mu \left(\ln \sqrt{|g_{\lambda\lambda}|} \right) \,, \\ \Gamma^{\lambda}{}_{\lambda\lambda} &= \partial_\lambda \left(\ln \sqrt{|g_{\lambda\lambda}|} \right) \,. \end{aligned} \tag{5}$$

Here, $\lambda \neq \nu \neq \mu \neq \lambda$ and repeated indices are *not* summed over. *(4 points)*

There is yet another way of calculating Christoffel symbols. To this end, consider curves in a manifold M,

$$\begin{aligned} x : \ (a,b) \subset \mathbb{R} &\longrightarrow M \\ \sigma &\longmapsto x(\sigma) \,, \end{aligned} \tag{6}$$

and define a functional F by

$$F[x] = \frac{1}{2} \int_a^b g_{\mu\nu}(x(\sigma)) \left(\frac{\mathrm{d}x^\mu(\sigma)}{\mathrm{d}\sigma} \right) \left(\frac{\mathrm{d}x^\nu(\sigma)}{\mathrm{d}\sigma} \right) \mathrm{d}\sigma \,. \tag{7}$$

c) Show that the Euler-Lagrange equation for F leads to the so called *geodesic equation*

$$\frac{\mathrm{d}^2 x^\mu(\sigma)}{\mathrm{d}\sigma^2} + \Gamma^{\mu}{}_{\nu\lambda} \left(\frac{\mathrm{d}x^\nu(\sigma)}{\mathrm{d}\sigma} \right) \left(\frac{\mathrm{d}x^\lambda(\sigma)}{\mathrm{d}\sigma} \right) = 0 \,. \tag{8}$$

From the geodesic equation one can unambiguously read of the Christoffel symbols. *(2 points)*

d) Explicitly find F for de Sitter space — with metric $\mathrm{d}s^2_{dS^4}$ as in eq. (4) — and calculate the Euler-Lagrange equation. From this identify the Christoffel symbols. *(4 points)*