Physikalisches Institut
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Theoretische Physik

Exercise 6
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Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/klemm/gr_15/

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Due on May 18, 2015

H 6.1 Proper time in general relativity and GPS *(17 points)*

A good approximation to the metric outside the surface of Earth is

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2(d\theta^2 + \sin\theta^2\, d\phi^2)\ , \tag{1}$$

with the Newtonian gravitational potential $\Phi = -GM/r$. Here, $G = 6.67 \cdot 10^{-11}\,\mathrm{m^3/kg/s^2}$ is Newton's gravitational constant and $M = 5.97 \cdot 10^{24}\,\mathrm{kg}$ is the mass of Earth. We will further need the radius of Earth, $R_0 = 6\,371\,\mathrm{km}$. The coordinates are chosen such that the spatial origin is located in the center of Earth and Earth rotates around the $\theta = 0$ axis.

a) For which observers is the coordinate time t the proper time? (*1 point*)

b) Consider a clock not moving relative to Earth (because it is resting on your desk). Keeping r and θ still arbitrary, calculate the time measured by this clock as a function of the elapsed coordinate time. You will see two effects, discuss these. (*4 points*)

c) Which clock runs faster: One resting on the surface of Earth or one on top of a tall building? Let us define a *second* by the requirement that a reference clock at $r = R_0$ and $\theta = \pi/2$ (equator) measures 24 h for one revolution of Earth. (*4 points*)

d) Solve for a geodesic corresponding to a circular orbit around the equator. In particular, find $d\phi/dt$. (*4 points*)

e) Now consider a GPS satellite orbiting at an altitude of 20 200 km above the surface of Earth around the equator. What is the time measured by a clock on this satellite needed for one complete orbit (relative to the also rotating earth)? Compare this to the time measured by the reference clock defined in item c). What are the absolute time difference and the relative deviation? (*4 points*)

H 6.2 Locally inertial frames *(8 points)*

Consider a Lorentzian manifold M with metric tensor g and a point $p \in M$. In this exercise we want to show that we can always find coordinates in which M looks like Minkowski space locally around p. We start with general $g_{\mu\nu}(p)$. Without loss of generality we can assume that the coordinates of p are zero.

a) Argue that there are coordinates in which $g_{\mu\nu}(p) = \eta_{\mu\nu}$. (*2 points*)

b) Change coordinates from x^μ to $x'^\mu = x^\mu + b^\mu{}_{\alpha\beta} x^\alpha x^\beta$. Show that $g'_{\mu\nu}(p) = \eta_{\mu\nu}$ still holds and find $b^\mu{}_{\alpha\beta}$ such that $\partial'_\alpha g'_{\mu\nu}(p) = 0$. This implies that all Christoffel symbols vanish and we have constructed a locally inertial frame. (*3 points*)

c) Which coordinate transformations are we now still allowed to perform such that the transformed frame is stays locally inertial? (*1 point*)

d) Do the constructed coordinates always coincide with the *Riemann normal coordinates* introduced in the lecture? Provide arguments for your answer. (*2 points*)

H 6.3 Explicit calculation of the Riemann tensor *(5 points)*

Consider the two-dimensional metric

$$\mathrm{d}s^2 = 2\mathrm{d}t^2 + t^2 \mathrm{d}\phi^2 \ . \tag{2}$$

a) Calculate the components of the Riemann tensor. You may want to think about how many independent components there are in two dimensions. (*3 points*)

b) Which surface has — embedded in $\mathbb{R}^3$ with the canonical Euclidean metric — the metric given in eq. (2)? (*2 points*)