Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 7
May 21, 2015
Summer term 2015

Exercises on General Relativity and Cosmology

Dr. Hans Jockers, Prof. Dr. Hans-Peter Nilles

http://www.th.physik.uni-bonn.de/klemm/gr_15/

–Comments–

This is a revised version. As compared to sheet handed out during the lecture, eq. (2) in H 7.1 b) and the formulation of H 7.2 e) have been corrected.

–Home Exercises–

Due on June 1, 2015

H 7.1 Hyperbolic spaces *(16 points)*

In the lecture and on the previous sheets you have seen examples of flat spaces, e.g. $\mathbb{R}^n$ or $\mathbb{M}^{1,n}$, as well as positively curved spaces, e.g. S^2 of radius R_{S^2} with constant Ricci scalar $R = 2/R_{S^2}^2$. Here we consider spaces with constant negative scalar curvature, which are called *hyperbolic*.

a) Calculate the Ricci scalar R associated to the metric

$$\mathrm{d}s^2 = a^2\big(\mathrm{d}t^2 + \cosh^2 t\,\mathrm{d}\phi^2\big) \tag{1}$$

and show that it takes the constant negative value $R = -2/a^2$. *(2 points)*

Hint: Recall the result of H 5.2 b).

b) Consider $\mathbb{M}^{1,2}$, whose metric is $\mathrm{d}s^2 = -\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, and embed into it the surface given by

$$x^2 - y^2 - z^2 = a^2\ . \tag{2}$$

Parameterize this surface and show that the induced metric equals (1). *(2 points)*

c) Calculate the Ricci scalar R associated to the metric

$$\mathrm{d}s^2 = f(r)^2\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \tag{3}$$

in terms of the function f. Set $R = -2/a^2$ and solve the differential equation for f. This will introduce an integration constant. For which values of this constant can the metric (3) be brought into the form (1)? Show this explicitly by finding the appropriate coordinate transformation. *(4 points)*

d) Consider a geodesic $\mathcal{C}$ parameterized by the arc length τ. Show that the quantity

$$g\left(\partial_\phi, \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\tau}\right) = c \tag{4}$$

is constant along the geodesic for the metric (1). *(4 points)*

Hint: There is no need to explicitly solve the geodesic equation.

e) Use eq. (4) and the fact that τ is the arc length to find $\mathrm{d}\phi/\mathrm{d}t$. *(2 points)*

f) Integration of $\mathrm{d}\phi/\mathrm{d}t$ gives the geodesics. Do this for the case $c = a$. *(2 points)*

H 7.2 Covariant derivative of tensors *(14 points)*
The Levi-Civita connection is the connection for which $\nabla_\mu \partial_\nu = \Gamma^\lambda{}_{\mu\nu}\partial_\lambda$ holds, where $\Gamma^\lambda{}_{\mu\nu}$ are the Christoffel symbols. This is the connection relevant for general relativity, as was shown in the lecture to follow from the physical postulates.

a) Show by an explicit calculation in components that the metric is covariantly constant, i.e. $\nabla g = 0$, with respect to the Levi-Civita connection. *(2 points)*

b) For a torsion free connection proof the differential Bianchi identity *(4 points)*

$$(\nabla_X R)(Y,Z)V + (\nabla_Z R)(X,Y)V + (\nabla_Y R)(Z,X)V = 0\ . \tag{5}$$

Hint: Do a coordinate free calculation.

c) Now specialize to the Levi-Civita connection. Write eq. (5) in components and deduce *(4 points)*

$$\begin{aligned} &\nabla_\mu G^{\mu\nu} = 0\ , \\ \text{with}\quad &G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\ , \quad \text{known as } \textit{Einstein tensor.} \end{aligned} \tag{6}$$

A *differential form* ω of order r, or short *r-form*, is a totally antisymmetric $(0,r)$-tensor. In terms of its components this means that $\omega_{\mu_1,\ldots\mu_r}$ is antisymmetric under interchange of any pair of indices. Note that for $r = 0$ and $r = 1$ this condition is trivial, thus we identify 0-forms as functions and 1-forms as covectors. The *exterior derivative* d assigns to an r-form ω an $(r+1)$-form $\mathrm{d}\omega$ whose components are

$$(\mathrm{d}\omega)_{\mu_1,\ldots,\mu_{r+1}} = \sum_{i=1}^{r+1}(-1)^{i-1}\partial_{\mu_i}\omega_{\mu_1,\ldots,\mu_{i-1},\mu_{i+1},\ldots,\mu_{r+1}}\,. \tag{7}$$

The exterior derivative of 0-forms coincides with the differential of functions, which is the reason for the notation.

d) Show that the components of $\mathrm{d}\omega$ indeed transform as those of an $(0, r+1)$-tensor under change of coordinates. *(2 points)*

e) Show that the *antisymmetrized* covariant derivative of an r-form with respect to a torsion free connection coincides with its exterior derivative. Note, however, that the exterior derivative is independent of the connection. *(2 points)*