Physikalisches Institut
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Theoretische Physik

Exercise 8
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Exercises on General Relativity and Cosmology

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–Home Exercises–

Due on June 8, 2015

H 8.1 True or false? *(9 points)*

Decide whether the following statements are true or false and provide arguments for your answer.

a) Consider a curve $\mathcal{C} : (a, b) \longrightarrow M$, $\lambda \longmapsto \mathcal{C}(\lambda)$ with $\nabla_{\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\lambda}} \left(\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\lambda}\right) \neq 0$ for some $\lambda \in (a, b)$. Then $\mathcal{C}((a, b)) \subset M$ can not be a geodesic. *(3 points)*

b) A curve $\mathcal{C} : (a, b) \longrightarrow M$, $\tau \longmapsto \mathcal{C}(\tau)$ is a time-like curve with $\nabla_{\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\lambda}} \left(\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\lambda}\right) = 0$. Then $\mathcal{C}$ always describes the motion of a freely moving particle parameterized by its proper time $\tau = \lambda$. *(3 points)*

c) Observer A and B live in a static universe with the metric $(i, j = 1, \ldots, 3)$

$$\mathrm{d}s^2 = -g(\vec{x})\mathrm{d}t^2 + g_{ij}(\vec{x})\mathrm{d}x^i\mathrm{d}x^j \ . \tag{1}$$

They move along the worldlines

$$\begin{aligned} \mathcal{C}_a &= \begin{cases} (t, \vec{x}_0) \ , & 0 \leq t \leq t_1 \\ (t, \vec{x}_0 + \vec{c}\,(t - t_1)) \ , & t_1 \leq t \leq t_2 \ , \\ (t, \vec{x}_0 + \vec{c}\,(t_2 - t_1)) \ , & t_2 \leq t \end{cases} \\ \mathcal{C}_B &= \begin{cases} (t, \vec{x}_0) \ , & 0 \leq t \leq t_3 \\ (t, \vec{x}_0 + \vec{c}\,(t - t_3)) \ , & t_3 \leq t \leq t_3 + t_2 - t_1 \\ (t, \vec{x}_0 + \vec{c}\,(t_2 - t_1)) \ , & t_3 + t_2 - t_1 \leq t \end{cases} \end{aligned} \tag{2}$$

for $0 < t_1 < t_2 < t_3$ and a smooth vector valued function c. Then the observers have aged by the same amount in the coordinate system time interval $(0, t)$ for $t > t_3 + t_2 - t_1$. *(3 points)*

Hint: The answers are not long, this is more about conceptional thinking.

H 8.2 Once time-like, always time-like *(4 points)*

Show that a geodesic which is time-like at one event will always be time-like. Proof similarly that a geodesic which is space-like (null) at one event will always be space-like (null).
Hint: Consider the norm of the tangent vector to the geodesic along the geodesic.

H 8.3 Variation of the Einstein-Hilbert term *(17 points)*

In this exercise we want to consider the variation of the Einstein-Hilbert term,

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N} \int \mathrm{d}^4x \sqrt{-g} R \,, \tag{3}$$

under the variation of the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. Denote $g = \det g_{\mu\nu}$.

a) For a diagonalizable matrix M show the identity $\ln(\det M) = \mathrm{tr}(\ln M)$. Note that the metric is a real symmetric matrix which implies diagonalizability. *(2 points)*

Hint: Diagonalizability means that there is a matrix S such that $M = S^{-1}DS$ for D diagonal. Further use $\ln M = \sum_{k=1}^{\infty}(-1)^{k+1}/k \cdot (M - \mathbb{1})^k$.

b) Show the formulas *(3 points)*

i) $\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\lambda}\delta g_{\rho\lambda}$,

ii) $\delta g = g g^{\mu\nu}\, \delta g_{\mu\nu} = -g g_{\mu\nu}\, \delta g^{\mu\nu}$.

Hint: Use the result of a) for ii).

Now we move to the variation of the Ricci tensor. For this recall the components of the Riemann tensor,

$$R^{\rho}{}_{\mu\lambda\nu} = \underbrace{\partial_\lambda \Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu}}_{\text{term 1}} - \underbrace{(\lambda \leftrightarrow \nu)}_{\text{term 2}} \ . \tag{4}$$

We start by taking the variation of the Riemann tensor with respect to a variation of the connection,

$$\Gamma^{\rho}_{\nu\mu} \to \Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\nu\mu} \ . \tag{5}$$

The variation $\delta\Gamma^{\rho}_{\nu\mu}$ should be thought of as some function of $\delta g_{\mu\nu}$. As we will see later on we do, however, not even need to know this relation.

Since the difference of two connections can be shown to be a tensor, we are allowed to take the covariant derivative of $\delta\Gamma^{\rho}_{\nu\mu}$,

$$\nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} \ . \tag{6}$$

c) Show that $\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu})$. *(4 points)*

Hint: The two terms on the right hand side of this formula to not seperately stem from the two terms in the Riemann tensor. The variations of term 1 and term 2 rather mix and then combine into the formula to be shown.

d) Deduce $\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu})$. *(1 point)*

e) Now show *(2 points)*

$$\begin{aligned}&\delta(\sqrt{-g}R) = \\ &- g^{\mu\rho}g^{\nu\lambda}\delta g_{\rho\lambda}R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}\big(\nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu})\big)\sqrt{-g} + \frac{1}{2}R\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \ .\end{aligned} \tag{7}$$

f) Let X^μ be the components of a vector. Show that *(3 points)*

$$\sqrt{-g}\,\nabla_\mu X^\mu = \partial_\mu\left(\sqrt{-g}X^\mu\right) \ . \tag{8}$$

g) Use covariant constancy of the metric and eq. 8 to deduce *(2 points)*

$$g^{\mu\nu}\big(\nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})\big)\sqrt{-g} = \partial_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\nu\mu}\sqrt{-g}) - \partial_\nu(g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}\sqrt{-g}) \ . \tag{9}$$

The second term in eq. 7 is thus a total divergence and for fields that vanish sufficiently rapid at infinity it does not give a contribution to the equations of motion. In this case we end up with

$$\delta S_{\mathrm{EH}} = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}Rg^{\mu\nu} - R^{\mu\nu}\right)\delta g_{\mu\nu} \ . \tag{10}$$