Exercises on General Relativity and Cosmology

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–Home Exercises–

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H 9.1 Electromagnetism in curved space-time *(30 points)*

In problem H 2.2 we have considered classical electrodynamics in Minkowski space. Here, we generalize this discussion to curved space-time. The action now reads

$$\begin{aligned} S_M = &-\sum_{i=1}^{N} \int \mathrm{d}\sigma_i \left(m_i \sqrt{-g_{\alpha\beta}(x_i(\sigma_i))\dot{x}_i{}^{\alpha}(\sigma_i)\dot{x}_i{}^{\beta}(\sigma_i)} - q_i\, A_\alpha(x_i(\sigma_i))\dot{x}_i{}^{\alpha}(\sigma_i) \right) \\ &- \frac{1}{4}\int \mathrm{d}^4x\, \sqrt{-g(x)} F_{\alpha\beta}(x) F^{\alpha\beta}(x)\ , \end{aligned} \tag{1}$$

in terms of particles of mass m_i and charge q_i, described by their trajectories x_i with components x_i^μ, and the electromagnetic potential with components A_μ. Note that indices are now raised and lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$. As usual, the field strength tensor is defined by $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$.

a) Starting from the action of electromagnetism in Minkowski space, argue why the action is generalized to (1). *(2 points)*

In the first part of this exercise we want to study how the equations of motions for particles and the electromagnetic field look in curved space-time.

b) Take the variation of S_M with respect to x_i^μ. Show that upon parameterizing in terms of the proper time τ_i, the equation of motion reads *(4 points)*

$$m_i\big(\ddot{x}_i{}^{\mu}(\tau_i) + \Gamma^{\mu}{}_{\lambda\rho}\dot{x}_i{}^{\lambda}(\tau_i)\dot{x}_i{}^{\rho}(\tau_i)\big) = q_i\, F^{\mu}{}_{\nu}(x_i(\tau_i))\dot{x}_i{}^{\nu}(\tau_i)\ . \tag{2}$$

c) Rewrite the term that couples the particles and the electromagnetic field by introducing the charge-current density *(2 points)*

$$j^\mu(x) = \sum_{i=1}^{N} q_i \int \mathrm{d}\sigma_i\, \delta^{(4)}(x - x_i(\sigma_i))\, \frac{\dot{x}_i{}^{\mu}(\sigma_i)}{\sqrt{-g(x)}}\ . \tag{3}$$

d) Take the variation of S with respect to A_μ to derive the inhomogenous Maxwell's equations *(3 points)*

$$\frac{1}{\sqrt{-g(x)}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g(x)}F^{\nu\mu}(x)\right) = j^\nu(x)\ . \tag{4}$$

e) What about the homogeneous Maxwell's equations? (*2 points*)

Let us now turn to charge conservation, which reads $\partial_\mu j^\mu = 0$ in Minkowski space. Its covariant generalization needs to be valid in every coordinate frame, thus we expect that the partial derivative is replaced by the covariant derivative so as to promote it to a tensorial equation.

f) Use the inhomogenous Maxwell's equations (4) to deduce the covariant charge conservation law, (*3 points*)

$$\nabla_\mu j^\mu(x) = 0 \ . \tag{5}$$

g) In fact, the conservation law (5) can be shown to hold identically, i.e. it can be proven without using the equations of motion. Since the definition of j^μ in eq. (3) involves the Delta-distribution, the equation has to be checked in the distributional sense. This means you have to show

$$\int \mathrm{d}^4x \sqrt{-g(x)}\,(\nabla_\mu j^\mu(x))\,\phi(x) = 0 \ , \tag{6}$$

where ϕ is an arbitrary smooth function with compact support. Consequently, partial integrations can be performed and all surface terms vanish. (*4 points*)

Now we promote the metric to a dynamical field by adding the Einstein-Hilbert term introduced in H 7.3 to the action (1). The full action then reads

$$S_{\text{full}} = S_{\text{EH}} + S_M \ . \tag{7}$$

Further, we define the energy momentum tensor by

$$\delta S_M = \frac{1}{2}\int \mathrm{d}^4x \sqrt{-g(x)} T^{\mu\nu}(x)\delta g_{\mu\nu}(x) \ , \tag{8}$$

which leads to Einstein's field equations,

$$G^{\mu\nu} = 8\pi G_N T^{\mu\nu} \ , \tag{9}$$

upon varying the full action with respect to the metric. For simplification we work in parameterization by the proper time from heron.

h) For S_M as in eq. (1) show that the energy momentum tensor reads (*4 points*)

$$\begin{aligned} T^{\mu\nu}(x) = &\sum_{i=1}^{N} m_i \int \mathrm{d}\tau_i \frac{\dot{x}_i{}^\mu(\tau_i)\dot{x}_i{}^\nu(\tau_i)}{\sqrt{-g(x)}}\delta^{(4)}(x - x_i(\sigma_i)) \\ &- \frac{1}{4}F_{\alpha\beta}(x)F^{\alpha\beta}(x)g^{\mu\nu}(x) - F^\mu{}_\lambda(x)F^{\nu\lambda}(x) \ . \end{aligned} \tag{10}$$

i) Show that the energy momentum tensor (10) is identically symmetric. (*1 point*)

j) For vanishing electromagnetic fields, $F_{\mu\nu} = 0$, prove the covariant version of energy momentum conservation,

$$\nabla_\mu T^{\mu\nu}(x) = 0 \ , \tag{11}$$

without using eq. (9). Again, this is to be understood in the distributional sense. In doing so you will need the equation of motion for the particles (2), the conservation does thus not hold identically. This is not surprising, since the conservation also follows from Noether's theorem in which the equations of motion are used. (*5 points*)