Physikalisches Institut
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Theoretische Physik

Exercise Sheet 1
(in class)
WS 2018/19

Group Theory

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http://www.th.physik.uni-bonn.de/klemm/grouptheory/index.php

Due date: discussed in the tutorial

1.1 Groups Theory in Physics

Discuss applications of group theory in physics that come to your mind.

1.2 Group Axioms

In class we defined a group G as follows.

Definition: A group G is a set with a binary operation $\cdot : G \times G \to G, (a, b) \mapsto a \cdot b$ such that:

- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for any $a, b, c \in G$ (associativity),
- there exists an element $e \in G$ with $e \cdot a = a \cdot e = a$ for any $a \in G$ (identity),
- and for all $a \in G$ there exists a' with $a \cdot a' = a' \cdot a = e$ (inverse).

a) We can weaken the definition of a group to the 'right sided axioms', i.e.,

A group G is a set with a binary operation $\cdot : G \times G \to G, (a, b) \mapsto a \cdot b$ such that

- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for any $a, b, c \in G$ (associativity),
- there exists an element $e \in G$ with $a \cdot e = a$ for any $a \in G$ (right identity),
- and for all $a \in G$ there exists a' with $a \cdot a' = e$ (right inverse).

Show that this latter definition of a group is equivalent to the former definition of the group given in class.

Hint: First show that $a \cdot a = a$ implies that $a = e$. Use this result to argue that a right inverse element is also a left inverse element. Finally, use the equality between the left and right inverse element to show that the left identity element is also a right identity element.

b) Let us examine the binary operation $\cdot : G \times G \to G, (a, b) \mapsto a \cdot b$ with 'mixed sided axioms', i.e.,

- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for any $a, b, c \in G$ (associativity),
- there exists an element $e \in G$ with $a \cdot e = a$ for any $a \in G$ (right identity),
- and for all $a \in G$ there exists a' with $a' \cdot a = e$ (left inverse).

Show that these axioms are not equivalent to the axioms defining a group by giving a counterexample.

1.3 Properties of the Inverse Elements

a) Show that for any two group elements a, b of a group G the inverse elements a^{-1} and b^{-1} obey the relation

$$(a \cdot b)^{-1} = b^{-1} \cdot a^{-1} \ .$$

b) Show that if all group elements a of a group G fulfill $a^2 = e$ then the group G is Abelian.

c) Show that if G is a finite group of even order that there is an element $a \neq e$ with $a^2 = e$.

1.4 Direct Product of Groups

Given two groups G_1 and G_2. Show that the (set theoretic) Cartesian product $G_1 \times G_2$ — i.e., the set of ordered pairs (g_1, g_2) with $g_1 \in G_1$ and $g_2 \in G_2$ — together with the binary operation

$$(g_1, g_2) \cdot (g_1', g_2') := (g_1 g_1', g_2 g_2')$$

for any elements $g_1, g_1' \in G_1$ and $g_2, g_2' \in G_2$ forms a group $G_1 \times G_2$, which is called the direct product group of G_1 and G_2.