Physikalisches Institut
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Theoretische Physik

Exercise Sheet 10
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Group Theory

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http://www.th.physik.uni-bonn.de/klemm/grouptheory/index.php

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10.1 Properties of Real Lie Groups

In this exercise we want to study the properties of three different real Lie groups.

i) Consider the orthogonal group $O(n, \mathbb{R}) = \mathrm{Aut}_{\langle\cdot,\cdot\rangle}(\mathbb{R}^n)$ with the canonical scalar product:

$$\langle x, y\rangle := x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \ ,$$

for $x, y \in \mathbb{R}^n$. Show that the columns and rows of any $M \in O(n, \mathbb{R})$ form an orthonormal basis in $\mathbb{R}^n$ and that $M^T = M^{-1}$.

Determine the possible values of the determinant of M and thence argue that $O(n, \mathbb{R})$ is not connected.

ii) Consider the unitary group $U(n) = \mathrm{Aut}_{\langle\cdot,\cdot\rangle}(\mathbb{C}^n)$ with the canonical Hermitian scalar product:

$$\langle x, y\rangle := \bar{x}_1 y_1 + \bar{x}_2 y_2 + \ldots + \bar{x}_n y_n \ ,$$

for $x, y \in \mathbb{C}^n$. Show that the columns of any $M \in U(n)$ form an orthonormal basis in $\mathbb{C}^n$ and that $M^\dagger = M^{-1}$ (where $M^\dagger := \bar{M}^T$).

Determine the possible values of the determinant of M.

Remark: $U(n)$ is connected but one can show that the range of the determinant implies that it is not simply connected.

iii) Consider the special orthogonal group $SO(1, 3; \mathbb{R}) = \mathrm{Aut}_{\langle\cdot,\cdot\rangle}(\mathbb{R}^n) \cap SL(n, \mathbb{R}^n)$ with the canonical bilinear pairing:

$$\langle x, y\rangle := -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 \ .$$

Show that $SO(1, 3; \mathbb{R})$ is neither compact nor connected.

(*10 pts*)

10.2 Vector Fields and the Lie Algebra

Prove proposition 3.7 (given in the lecture) which states:

The space $\mathcal{H}(M)$ of smooth vector fields on a manifold M is a (real) vector space, which together with the Lie bracket forms a Lie algebra over $\mathcal{H}(M)$.

(*10 pts*)