Physikalisches Institut
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Theoretische Physik

Exercise Sheet 11
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Group Theory

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http://www.th.physik.uni-bonn.de/klemm/grouptheory/index.php

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10.1 Lie Groups and Lie Algebras

In this exercise we want to study the relationship among Lie algebras of different Lie groups. In particular, we want to analyze Lie algebra isomorphisms and their lifts to the Lie groups.

i) Determine the vector spaces underlying the Lie algebras of $\mathrm{SO}(n,\mathbb{R}), \mathrm{U}(n)$ and $\mathrm{SU}(n)$. Confirm explicitly that the Lie brackets close on the elements of the respective Lie algebras. Deduce the dimensions of $\mathfrak{so}(n,\mathbb{R}), \mathfrak{u}(n)$ and $\mathfrak{su}(n)$.

ii) Determine the Lie algebra of the complex group $\mathrm{Sp}(2n,\mathbb{C})$ and compute its complex dimension. Remember $\mathrm{Sp}(2n,\mathbb{C})$ is defined as the automorphism group of the non-degenerate skew-symmetric pairing

$$\langle\cdot,\cdot\rangle : \ \mathbb{C}^{2n}\times\mathbb{C}^{2n} \longrightarrow \mathbb{C} \ , \quad (x,y)\longmapsto \langle x,y\rangle := x^t\Omega y \ , \quad \text{with } \Omega = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix} \ .$$

iii) We define the compact symplectic group $\mathrm{Sp}(n) := \mathrm{Sp}(2n,\mathbb{C})\cap\mathrm{U}(2n)$ which is a real group. Determine with the help of i) and ii) the Lie algebra of $\mathrm{Sp}(n)$ and its dimension.

iv) Spell out the Lie algebras of $\mathrm{SO}(3,\mathbb{R}), \mathrm{SU}(2)$ and $\mathrm{Sp}(1)$ explicitly and show that they are isomorphic.

We now want to study the relations among the Lie groups $\mathrm{SO}(3,\mathbb{R}), \mathrm{SU}(2)$ and $\mathrm{Sp}(1)$.

v) Show that the Lie groups $\mathrm{SU}(2)$ and $\mathrm{Sp}(1)$ are the same.
Hint: First, show that the most general element of $\mathrm{U}(2)$ can be written as

$$\begin{pmatrix} \alpha e^{i\varphi} & \beta \\ -\bar\beta e^{i\varphi} & \bar\alpha \end{pmatrix} \quad \text{with } |\alpha|^2+|\beta|^2 = 1 \text{ and } \varphi\in\mathbb{R} \ .$$

Then intersect these matrices with $\mathrm{Sp}(2,\mathbb{C})$ to construct a general element of $\mathrm{Sp}(1)$.

vi) Consider a general group element $\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$ of $\mathrm{SU}(2)$ to argue that as a manifold $\mathrm{SU}(2)$ is the 3-sphere S^3 (embedded in $\mathbb{R}^4$).

vii) We consider the underlying vector space of the Lie algebra $\mathfrak{su}(2) \simeq \mathbb{R}^3$ and define the scalar product

$$\langle \cdot, \cdot \rangle : \ \mathfrak{su}(2) \times \mathfrak{su}(2) \longrightarrow \mathbb{R} , \quad (x, y) \longmapsto \langle x, y \rangle := \frac{1}{2} \mathrm{tr}(x y^\dagger) \ .$$

Show that this scalar product is the canonical scalar product of $\mathbb{R}^3$ upon identifying the basis vectors $(e_k)_{k=1,2,3}$ with $(i\sigma_k)_{k=1,2,3}$ in terms of the Pauli matrices σ_k.

viii) Show that for any $U \in \mathrm{SU}(2)$ the map

$$\varphi_U : \ \mathfrak{su}(2) \longrightarrow \mathfrak{su}(2) , \quad x \longmapsto \varphi_U(x) := U x U^{-1}$$

is well defined and preserves the scalar product. This implies that we have a map

$$\Phi : \ \mathrm{SU}(2) \longrightarrow \mathrm{Aut}_{\langle \cdot, \cdot \rangle}(\mathfrak{su}(2)) \simeq \mathrm{O}(3, \mathbb{R}) , \quad U \longmapsto \varphi_U \ .$$

ix) Show that the image $\Phi(U) \in \mathrm{O}(3, \mathbb{R})$ for a general $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(2)$ is given by

$$\Phi(U) = \begin{pmatrix} \mathrm{Re}(\alpha^2 - \beta^2) & \mathrm{Im}(\alpha^2 + \beta^2) & -2\mathrm{Re}(\alpha\beta) \\ -\mathrm{Im}(\alpha^2 - \beta^2) & \mathrm{Re}(\alpha^2 + \beta^2) & 2\mathrm{Im}(\alpha\beta) \\ 2\mathrm{Re}(\alpha\bar{\beta}) & 2\mathrm{Im}(\alpha\bar{\beta}) & |\alpha|^2 - |\beta|^2 \end{pmatrix} .$$

The image of Φ is actually in $\mathrm{SO}(3, \mathbb{R})$ because $\det(\Phi(U)) = 1$ (which you do not need to verify).

Hint: Consider the images of the basis elements $i\sigma_k \in \mathrm{SU}(2)$ with respect to the map φ_U defined in viii).

x) We now want to show further that Φ. $\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3, \mathbb{R})$ is surjective. Recall that any matrix $M \in \mathrm{SO}(3, \mathbb{R})$ can be obtained from the composition of rotations around the 3 coordinate axes generated by $\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{pmatrix}$. Show that these generators are in the image of Φ, which implies that Φ is surjective (because it is a Lie group homomorphism).

xi) Show that the kernel of Φ is the subgroup $\mathbb{Z}_2$ of $\mathrm{SU}(2)$ (which is also the center of $\mathrm{SU}(2)$).

(20 pts)

Summary of the results

1. $\mathrm{SO}(3, \mathbb{R}), \mathrm{SU}(2)$ and $\mathrm{Sp}(1)$ are real Lie groups of dimension 3. They have isomorphic Lie algebras, i.e. $\mathfrak{so}(3, \mathbb{R}) \simeq \mathfrak{su}(2) \simeq \mathfrak{sp}(1)$.

2. $\mathrm{SU}(2)$ and $\mathrm{Sp}(1)$ are isomorphic Lie groups.

3. $\mathrm{SO}(3, \mathbb{R})$ and $\mathrm{SU}(2)$ are <u>not</u> isomorphic Lie groups. Instead they fit into the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3, \mathbb{R}) \longrightarrow 0$$

which makes $\mathrm{SU}(2)$ a central extension of $\mathrm{SO}(3, \mathbb{R})$ by $\mathbb{Z}_2$.
This implies that as a manifold $\mathrm{SO}(3, \mathbb{R})$ is $S^3/\mathbb{Z}_2$, where antipodal points of S^3 are identified. Topologically, $\mathrm{SO}(3, \mathbb{R})$ is not simply-connected, and $\mathrm{SU}(2) \simeq S^3$ is called a double cover of $\mathrm{SO}(3, \mathbb{R}) \simeq S^3/\mathbb{Z}_2$.