Physikalisches Institut
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Theoretische Physik

Exercise Sheet 12
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Group Theory

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http://www.th.physik.uni-bonn.de/klemm/grouptheory/index.php

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12.1 Direct Sum, Tensor Product and Dual Representation

Let G be a Lie group with two representations

$$\rho_V : G \longrightarrow \mathrm{GL}(V,\mathbb{C}) \quad \text{and} \quad \rho_W : G \longrightarrow \mathrm{GL}(W,\mathbb{C}).$$

i) Consider the direct sum representation $\rho_{V\oplus W} : G \longrightarrow \mathrm{GL}(V,\mathbb{C})$. Show that the Lie algebra homomorphism $(\mathrm{d}\rho_{V\oplus W})_e : \mathfrak{g} \longrightarrow \mathfrak{gl}(V\oplus W,\mathbb{C})$ is given by

$$(\mathrm{d}\rho_{V\oplus W})_e = (\mathrm{d}\rho_V)_e \oplus (\mathrm{d}\rho_W)_e \ .$$

ii) Consider the tensor product $\rho_{V\otimes W} = \rho_V \otimes \rho_W : G \longrightarrow \mathrm{GL}(V\otimes W,\mathbb{C})$. Show that

$$(\mathrm{d}\rho_{V\otimes W})_e = ((\mathrm{d}\rho_V)_e \otimes \mathbb{1}_W) \oplus (\mathbb{1}_V \otimes (\mathrm{d}\rho_W)_e) \ .$$

iii) Consider the dual representation ρ_{V^*} and show that

$$(\mathrm{d}\rho_{V^*})_e = -(\mathrm{d}\rho_V)^* : \mathfrak{g} \longrightarrow \mathfrak{gl}(V^*,\mathbb{C}) \ .$$

<u>*Hint*</u>: Represent ρ_V in matrix form, namely $\rho_V : G \longrightarrow \mathrm{GL}(n,\mathbb{C}), \quad g \longmapsto R_V(g)$ with $n = \mathrm{dim}V$. Recall $\rho_{V^*} : G \longrightarrow \mathrm{GL}(n,\mathbb{C}),\ g \longmapsto R_{V^*}(g) = R_V(g^{-1})^T$.

(5 pts)

12.2 The Exponential Map

We want to evaluate the exponential map for some elements of the Lie algebra $\mathfrak{su}(2)$ and $\mathfrak{so}(3,\mathbb{R})$ for the matrix Lie groups SU(2) and SO(3, ℝ).

i) Evaluate the exponential map:

$$\exp : \mathfrak{su}(2) \to \mathrm{SU}(2) \subset \mathbb{C}^{2\times 2} \ ,$$

for the Lie algebra element $\begin{pmatrix} 0 & i\theta \\ i\theta & 0 \end{pmatrix} = \theta \cdot i\sigma_1$, where $\theta \in \mathbb{R}$.

ii) Evaluate the exponential map:

$$\exp : \mathfrak{so}(3,\mathbb{R}) \to \mathrm{SO}(3,\mathbb{R}) \subset \mathbb{R}^{3\times 3} \ ,$$

for the Lie algebra element $\begin{pmatrix} 0 & 2\theta & 0 \\ -2\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \theta \cdot \tilde{\tau}_1$, where $\theta \in \mathbb{R}$.

iii) The Lie algebra isomorphism of ex.11.1 maps $\begin{pmatrix} 0 & i\theta \\ i\theta & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & 2\theta & 0 \\ -2\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Compare the results of (i) and (ii) in the light of the outcome of ex.11.1.

12.3 Derived Series and Semi-Simple Lie Algebras

Consider a Lie algebra $\mathfrak{g}$ with an ideal $\mathfrak{h}$. The derived series $\mathcal{D}^k\mathfrak{h}$ is recursively defined by:

$$\mathcal{D}\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \quad , \quad \mathcal{D}^k\mathfrak{h} = [\mathcal{D}^{k-1}\mathfrak{h}, \mathcal{D}^{k-1}\mathfrak{h}] \ ,$$

where $[\mathfrak{a}, \mathfrak{b}]$ denotes the subalgebra of $\mathfrak{g}$, which is spanned by $[a, b]$ for any $a \in \mathfrak{a}$ and any $b \in \mathfrak{b}$. Show that $\mathcal{D}^k\mathfrak{h}$ is an ideal in $\mathfrak{g}$ for any $k \geq 1$.

<u>*Remark*:</u> The ideal $\mathfrak{h}$ is called solvable if $\mathcal{D}^k\mathfrak{h} = 0$ for some k. An alternative (but equivalent) definition of a semi-simple Lie algebra $\mathfrak{g}$ is given by saying that $\mathfrak{g}$ has no solvable ideals.

(5 pts)

12.4 A Double Cover

Let $A, B \in \mathbb{C}^{2\times 2}$ be elements in the space of 2×2 matrices over the field $\mathbb{C}$ equipped with the form:

$$Q(A, B) = \frac{1}{2}\mathrm{tr}(A \ \mathrm{adj}(B)) \ ,$$

where adj(M) denotes the classical adjoint of a matrix M and takes for 2×2 matrices the form, $\mathrm{adj}\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$.

i) Show that the form $Q(\cdot, \cdot)$ is symmetric and non-degenerate.

ii) Argue that the space of maps:

$$f_{g,h} : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2} \ , \ A \mapsto g \cdot A \cdot h^{-1} \ ,$$

for any $g, h \in \mathrm{SL}(2, \mathbb{C})$ can be interpreted as elements of $\mathrm{SO}(4, \mathbb{C})$.

Furthermore, argue that the map:

$$\varphi : \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(4, \mathbb{C}) \ , \ (g, h) \mapsto f_{g,h} \ ,$$

is a $2:1$ Lie group homomorphism.

<u>*Remark*:</u> In exercise 11.1 we have seen that SU(2) is the double cover of $\mathrm{SO}(3, \mathbb{R})$. The constructed Lie group homomorphism φ shows similarly that $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ is the double cover of $\mathrm{SO}(4, \mathbb{C})$.