Physikalisches Institut
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Theoretische Physik

Exercise Sheet 13
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WS 2018/19

Group Theory

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`http://www.th.physik.uni-bonn.de/klemm/grouptheory/index.php`

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13.1 Representations of the Lie Algebras $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{su}(2)$, $\mathfrak{so}(3,\mathbb{R})$ and their Lie Groups

i) Let $V^{(n)}$ be the finite dimensional irreducible representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, which decomposes into $V^{(n)} = \bigoplus_{l=0}^{n} V_{n-2l}$ in terms of the eigenspaces V_{n-2l} with respect to the Cartan generator H. Show that the representations $V^{(n)}$ and $\text{Sym}^n V^{(1)}$ are isomorphic for all $n \geq 1$.

ii) Show that the irreducible representations $V^{(n)}$ of $\mathfrak{sl}(2,\mathbb{C})$ restrict to irreducible representations $\tilde{V}^{(n)}$ of the real lie algebra $\mathfrak{su}(2)$ and hence of $\mathfrak{so}(3,\mathbb{R})$.
<u>*Hint*:</u> $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C}$

<u>*Remark*:</u> Note that any irreducible finite dimensional Lie algebra representation of $\mathfrak{su}(2)$ is of the form $\tilde{V}^{(n)}$.

iii) Argue that any representation $\tilde{V}^{(n)}$ of the Lie algebra $\mathfrak{su}(2)$ lifts to a representation of the Lie group SU(2).

iv) Show that the Lie algebra representation $\tilde{V}^{(n)}$ of $\mathfrak{so}(3,\mathbb{R})$ lifts only to a Lie group representation of $\text{SO}(3,\mathbb{R})$ if n is even.
<u>*Hint*:</u> Recall the results of exercise 11.1 and consider the action of the center of the Lie group SU(2) on the Lie group representation obtained from $\tilde{V}^{(n)}$.

(*10 pts*)

13.2 Tensor Products of Lie Algebra Representation $V^{(n)}$ of $\mathfrak{sl}(2,\mathbb{C})$

i) Decompose the tensor products

$$V^{(1)} \otimes V^{(1)} \quad \text{and} \quad V^{(5)} \otimes V^{(2)}$$

of Lie algebra representations into their irreducible summands.
<u>*Hint*:</u> Recall the result of exercise 12.1.

ii) Show that in general we have for $n \geq m$ the decomposition

$$V^{(n)} \otimes V^{(m)} \simeq V^{(n+m)} \oplus V^{(n+m-2)} \oplus \ldots \oplus V^{(n-m)} \ .$$

<u>*Remark*:</u> Accordingly to the previous exercise 13.1, the representation theory of $\mathfrak{sl}(2,\mathbb{C})$ carries over to $\mathfrak{su}(2)$. Therefore, this decomposition of these tensor products becomes relevent in the context of spin-spin coupling in atomic physics.

(*10 pts*)