Physikalisches Institut
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Theoretische Physik

Exercise Sheet 2
10/10/2018
WS 2018/19

Group Theory

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http://www.th.physik.uni-bonn.de/klemm/grouptheory/index.php

Due date: 10/17/2018

2.1 The Symmetric Group S_n

Let us consider the symmetric group S_n. The group elements of S_n are the permutations of n elements, which we describe in terms of bijective functions $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. The composition of such bijective functions corresponds to the binary group operation of S_n.

Since the permutation σ is a function on a finit set, it can conveniently be described by listing the elements and their images in two rows

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \ .$$

a) Demonstrate that the symmetric group S_n (for $n \geq 3$) is non-Abelian. *(1 pt)*

Group elements of the symmetric group S_n can also be presented in terms of the cycle notations. A cycle $\mathcal{C} = (x_1, \ldots, x_k)$ with $x_i \in \{1, \ldots, n\}$ and $x_i \neq x_j$ for all $i \neq j$ is the permutation $\sigma_{\mathcal{C}}$ with $\sigma_{\mathcal{C}}(x_i) = x_{i+1}$ (where we set $x_{k+1} \equiv x_1$) and $\sigma_{\mathcal{C}}(x) = x$ for any other $x \notin \{x_1, \ldots, x_k\}$. Since the order of S_n is finite, it is not so difficult to see that any permutation σ can be written as a finite composition of disjoint cycle permutations, i.e., $\sigma = (\mathcal{C}_1, \ldots, \mathcal{C}_\ell) = \mathcal{C}_1 \circ \ldots \circ \mathcal{C}_\ell$. For instance we have

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 6 & 5 & 2 \end{pmatrix} = ((1,3),(2,4,6)) \ .$$

b) Argue that the permutation element in the cycle notation does not depend on the order of the cycles. *(2 pts)*

c) Consider the permutation element $\alpha = (1,5) \circ (1,2,6) \circ (1,3,2) \circ (1,4,5)$ and $\beta = (3,6) \circ (1,2,4) \circ (2,4,3) \circ (2,3) \circ (1,5)$ in S_6. Determine α, β, α^{-1}, β^{-1}, $\beta\alpha$, and $\alpha\beta$ in cycle notation. *(3 pts)*

d) Given any permutation $\alpha \in S_n$, describe and prove a quick method for determining α^{-1} in cycle notation. *(2 pts)*

2.2 Abelian Groups

Show that the following conditions on a group are equivalent:

(i) G is Abelian,

(ii) $(ab)^2 = a^2 b^2$ for all $a, b \in G$,

(iii) $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$,

(iv) $(ab)^n = a^n b^n$ for all $n \in \mathbb{Z}$ and all $a, b \in G$,

(v) $(ab)^n = a^n b^n$ for three consecutive integers n and all $a, b \in G$.

(10 pts)

2.3 Left- and Right Multiplication

For a group G, let us define the maps $l_a : G \to G,\ x \mapsto a \cdot x$ and the maps $r_a : G \to G, x \mapsto x \cdot a$ for any $a \in G$. These maps are called left an right multiplication maps with a, respectively. Show that for any a these maps are bijective. *(2 pts)*