Group Theory

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http://www.th.physik.uni-bonn.de/klemm/grouptheory/index.php

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3.1 Abelian Groups and Automorphisms

Show that a group G is abelian if and only if the map

$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

is a group automorphism.
<u>*Hint*</u>: Remember that an automorphism is a group homomorphism between G and itself which is also bijective.

(3 pts)

3.2 Subgroups of $\mathbb{Z}$

If n is a fixed integer, then we can define the set $n\mathbb{Z} = \{kn|k \in \mathbb{Z}\} \subset \mathbb{Z}$.

a) Show that $n\mathbb{Z}$ is an additive subgroup of $\mathbb{Z}$.

(2 pts)

b) Show that $n\mathbb{Z}$ is isomorphic to $\mathbb{Z}$.

(1 pt)

3.3 Conjugacy Classes and Normal Subgroups

In this exercise we want to analyse the relation between conjugacy classes and normal subgroups. Therefore, we recall the definition of an equivalence relation.

> <u>*Definition*</u>: A binary relation $\sim$ on a set X is called an equivalence relation if it satisfies the following properties:
>
> i) $a \sim a$ *Reflexivity*
>
> ii) $a \sim b$ if and only if $b \sim a$ *Symmetry*
>
> iii) $a \sim b$ and $b \sim c$ then also $a \sim c$ *Transitivity*.

If G is a group we can define conjugation by an element $h \in G$ in the following way

$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto hgh^{-1} \ . \end{aligned}$$

a) Show that conjugation defines an equivalence relation, i.e., $g \sim h$ if and only if there exists $x \in G$ such that $g = xhx^{-1}$.

(2 pts)

A subgroup $H \subset G$ is called normal if it is self-conjugate which means that $gHg^{-1} = H$ for all $g \in G$.

b) If H is a normal subgroup of G then show that it can be written as a union of conjugacy classes of G.

(2 pts)

c) Prove that all subgroups of Abelian groups are normal.

(1 pt)

d) Let G be a finite group and H a subgroup of it such that it contains half the elements of G, i.e., $|G|/|H| = 2$. Show that H is a normal subgroup.
Hint: Decompose the group G into cosets of H.

(3 pts)

3.4 The Center of a Group

If G is a group, the we can define the center $Z(G)$ by

$$Z(G) = \{h \in G | hg = gh \text{ for all } g \in G\} \ .$$

a) Show that the center $Z(G)$ is an abelian subgroup of G

(2 pts)

b) Moreover, show that the center $Z(G)$ is a normal subgroup of G

(1 pt)

c) Let S_n be the permutation group of n elements. Prove that for $n > 2$ the center of S_n is the identity subgroup.

(3 pts)