Physikalisches Institut
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Theoretische Physik

Exercise Sheet 5
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Group Theory

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http://www.th.physik.uni-bonn.de/klemm/grouptheory/index.php

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5.1 The Quaternion Group

In the following we want to analyze the so called quaternion group Q_8. We define the group Q_8 as the group under ordinary matrix multiplication generated by the complex matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, with $i^2 = -1$.

i) Show that Q_8 is a non-abelian group of order 8.
Hint: It is useful to show that $A^4 = B^4 = \mathbb{1}_{2\times 2}$ and $BA = A^3 B$.

Now, we want to define the quaternions more abstractly. Consider the set $G = \{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication given by $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$ and the usual rules for multiplying by ± 1.

ii) Show that G is a group that is isomorphic to the quaternion group Q_8.

The order of an element $g \in G$ is defined as the smallest positive integer n such that $g^n = e$. For finite groups the order of any element is finite.

iii) Prove that the dihedral group D_4 is not isomorphic to Q_8.
Hint: Consider elements of order 2.

iv) Prove that the following short exact sequence does not split

$$1 \longrightarrow N \longrightarrow Q_8 \longrightarrow Q_8/N \longrightarrow 1 \ ,$$

where N is the center of the quaternion group.

(2+2+2+2 pts)

5.2 Transpositions

On the second exercise sheet we analyzed the symmetric group S_n in terms of cycles. In this exercise we want to use transpositions.

- A transposition is defined as a 2-cycle, i.e. $\tau = (x_1, x_2)$ meaning that the two elements x_1 and x_2 of $\{1, 2, ..., n\}$ exchange positions.

- One can show that every permutation $\sigma \in S_n$ can be written as a product of (not necessarily disjoint) transpositions.
- Furthermore, a permutation $\sigma \in S_n$ is even (resp. odd) if σ can be written as a product of an even (resp. odd) number of transpositions. In particular, it can be shown that a permutation cannot be both even and odd.

i) let α, β be even and γ, δ be odd permutations respectively. Show that

$$\alpha \cdot \beta \text{ is even}, \quad \gamma \cdot \delta \text{ is even} \quad \text{and} \quad \alpha \cdot \gamma \text{ is odd}.$$

ii) For each $n \geq 2$, let A_n be the set of all even permutations. Show that it is a normal subgroup of S_n.

iii) Determine the index of A_n in S_n and the order of A_n.

iv) Show that $N = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4 contained in A_4 such that $S_4/N \cong S_3$ and $A_4/N \cong \mathbb{Z}_3$.

(1+1+2+4 pts)

5.3 Classification of finite groups

In this exercise we want to classify all finite groups of order 4.

i) Show by using Lagrange's theorem that every finite group of order 4, which is not cyclic, must contain the identity element and three other elements of order 2.

ii) Prove that any non-cyclic finite group of order 4 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Putting both parts together we have shown that up to isomorphisms all finite groups of order 4 are given by the cyclic group $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(2+2 pts)