Group Theory

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6.1 Direct Sums and Tensor Products

Consider two matrices, $A \in \mathbb{C}^{p\times q}, B \in \mathbb{C}^{r\times s}$. For matrices, the direct sum is defined as

$$A \oplus B \in \mathbb{C}^{(p+r)\times(q+s)} \quad \text{with} \quad (A\oplus B)_{ij} = \begin{cases} A_{ij}, & i \le p \text{ and } j \le q \\ B_{(i-p)(j-q)}, & i > p \text{ and } j > q \\ 0, & \text{else} \end{cases}$$

and the tensor product is defined as

$$A \otimes B \in \mathbb{C}^{pr\times qs} \quad \text{with} \quad (A\otimes B)_{(ik)(jl)} = A_{ij}B_{kl} \ .$$

In block matrix form they can be visualized as:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \ , \quad A\otimes B = \begin{pmatrix} A_{11}B & \dots & A_{1q}B \\ \vdots & \ddots & \vdots \\ A_{p1}B & \dots & A_{pq}B \end{pmatrix} \ .$$

i) Show that:

$$(A\oplus B)^T = A^T \oplus B^T \ , \qquad \overline{A\oplus B} = \bar{A}\oplus\bar{B} \ ,$$
$$(A\otimes B)^T = A^T \otimes B^T \ , \qquad \overline{A\otimes B} = \bar{A}\otimes\bar{B} \ ,$$

where $\bar{}$ means complex conjugation, i.e. $(\bar{A})_{ij} = \bar{A}_{ij}$.

ii) Show that, if dimensions match, the following is true:

$$(A\oplus B)(C\oplus D) = AC \oplus BD \ , \quad (A\otimes B)(C\otimes D) = AC\otimes BD \ .$$

iii) Let $A \in \mathbb{C}^{m\times m}, B\in\mathbb{C}^{n\times n}$. Prove that:

$$\mathrm{tr}(A\oplus B) = \mathrm{tr}A + \mathrm{tr}B \ , \qquad \det(A\oplus B) = \det A \cdot \det B \ ,$$
$$\mathrm{tr}(A\otimes B) = \mathrm{tr}A\cdot\mathrm{tr}B \ .$$

iv) Let again $A \in \mathbb{C}^{m\times m}, B\in\mathbb{C}^{n\times n}$. Prove that:

$$\det(A\otimes B) = (\det A)^n \cdot (\det B)^m \ .$$

v) Given two vector spaces V, W, each vector in $V\otimes W$ can be represented by a $\mathrm{dim}V\times\mathrm{dim}W$ matrix. Show that the vectors of the form $v\otimes w \in V\otimes W$ correspond to rank one matrices.

(5 pts)

6.2 Characters of the Exterior Product and the Symmetric Product

Let V be a finite complex vector space of dimension n with a basis given by $\{e_1, e_2, \ldots, e_n\}$. Then the tensor product $V \otimes V$ has a basis $\{(e_i \otimes e_j)_{i,j=1,2,\ldots,n}\}$.
Let $e_i \wedge e_j \coloneqq \frac{1}{2}[(e_i \otimes e_j) - (e_j \otimes e_i)]$ and $e_i \vee e_j \coloneqq \frac{1}{2}[(e_i \otimes e_j) + (e_j \otimes e_i)]$ such that we can define

$$\bigwedge^2 V \coloneqq \langle e_i \wedge e_j \rangle_{1 \leq i < j \leq n} \ , \quad \text{and} \quad \mathrm{Sym}^2 V \coloneqq \langle e_i \vee e_j \rangle_{1 \leq i \leq j \leq n} \ .$$

i) Let V be a representation of the group G. Show that $\bigwedge^2 V$ and $\mathrm{Sym}^2 V$ are subrepresentations of $V \otimes V$ such that $\bigwedge^2 V \oplus \mathrm{Sym}^2 V = V \otimes V$.

ii) Show that the characters of $\bigwedge^2 V$ and $\mathrm{Sym}^2 V$ are respectively given by

$$\chi_{\wedge^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right) \ , \quad \chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right) .$$

Hint: Consider the eigenvalues of the group action for a fixed element $g \in G$ on V and determine the eigenvalues of the group action of G on $\bigwedge^2 V$ of $\mathrm{Sym}^2 V$.

(5 pts)

6.3 Characters and the Characteristic Polynomial

Let ρ_V be a finite complex representation of G, i.e. $\rho_V : G \longrightarrow \mathrm{GL}(V, \mathbb{C})$. We want to show that the characteristic polynomial P of the map $g \coloneqq \rho_V(g) : V \longrightarrow V$ for a fixed $g \in G$ is determined by the character $\chi_V : G \longrightarrow \mathbb{C}$. Work out explicitly for 2 and 3 dimensional representations the formulas for the characteristic polynomials in terms of their respective characters.

Remark: It can be shown using the theory of symmetric polynomials that the above relations can be extended for general n dimensional representations.

(4 pts)

6.4 Conjugacy Classes of S_n

In this exercise we want to show that there is a one-to-one correspondence between the conjugacy classes of S_n (see exercise 3.3) and partitions of n. Recall that a partition of n is a way of writing n as a sum of positive integers. Two partitions are equal if they only differ in the order of their summands. E.g. $1+1+1, 2+1, 3$ are all partitions of 3.
Remember that any element of S_n can be written in terms of disjoint cycles (see exercise 2.1).

i) Show that for any $\pi \in S_n$ and any cycle $\mathcal{C} = (x_1, ..., x_s)$, with $x_i \in \{1, ..., n\}$ and $x_i \neq x_j$ for all $i \neq j$

$$\pi \mathcal{C} \pi^{-1} = (\pi(x_1), ..., \pi(x_s)) =: \pi(\mathcal{C}) \ .$$

ii) Show that

$$\pi(\mathcal{C}_1 \circ \mathcal{C}_2 \circ \ldots \circ \mathcal{C}_r)\pi^{-1} = (\pi(\mathcal{C}_1) \circ \pi(\mathcal{C}_2) \circ \ldots \circ \pi(\mathcal{C}_r)) \ .$$

iii) Argue that there is a one-to-one correspondence between partitions of n and conjugacy classes of S_n.

iv) Work out all conjugacy classes of S_5.

(6 pts)