Physikalisches Institut
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Theoretische Physik

Exercise Sheet 8
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WS 2018/19

Group Theory

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http://www.th.physik.uni-bonn.de/klemm/grouptheory/index.php

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8.1 Irreducible Representations of Q_8

We want to work out all irreducible representations of the Quaternion group Q_8 defined in exercise 5.1.

i) Determine the number of conjugacy classes of Q_8 and the number of elements they contain.

ii) Check that $\mathbb{Z}_2$ is a normal subgroup of Q_8. Prove that $Q_8/\mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

iii) Determine the irreducible representations of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Lift these representations to irreducible representations of Q_8.

iv) Show that apart from the irreducible representations lifted from $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ there is one more irreducible representation of Q_8. Determine its dimension and character.

v) Classify these representations into real, quaternionic and complex $\mathbb{C}$-representations.

(*8 pts*)

8.2 Real $\mathbb{C}$-Representations of a Group

In this exercise we want to look at properties of real $\mathbb{C}$-representations.

i) Show that all characters of a finite group are real if and only if every element is conjugate to its inverse.

ii) Argue that all characters of the permutation group S_n are real.
Hint: Recall the results of exercise 6.4 .

iii) Show explicitly for S_4 that all representations are real $\mathbb{C}$-representations.
Remark: All irreducible $\mathbb{C}$-representations of S_n are real.

iv) Decide which irreducible representations of A_4 are real, quaternionic and complex $\mathbb{C}$-representations.

The character tables of S_4 and A_4 are listed at the end of the sheet. For an explicit calculation of the character tables refer to exercise sheet 7.

(*8 pts*)

8.3 Complex $\mathbb{C}$-Representations of a Group

In this exercise we want to show that if G is a group of odd order, then any non-trivial irreducible $\mathbb{C}$-representation is complex. Thus let G be a group of odd order, then:

i) Show that any $g \in G$ has odd order, i.e. $g^{2k+1} = e$ for some $k \in \mathbb{Z}$. Conclude that any element in G can be written as a square of another element.

ii) Let V be an irreducible $\mathbb{C}$-representation. Show that:

$$\sum_{g \in G} \chi_V(g) = \sum_{g \in G} \chi_V(g^2)$$

<u>*Hint*:</u> Use the conclusion of part i).

iii) Argue that if V is a non-trivial irreducible representation then it must be a complex $\mathbb{C}$-representation.
<u>*Hint*:</u> Use the orthonormality of irreducible characters.

(4 pts)

	[(1)]	[(12)]	[(12)(34)]	[(123)]	[(1234)]
#elements	1	6	3	8	6
$\mathbf{1}_T$	1	1	1	1	1
$\mathbf{1}_A$	1	-1	1	1	-1
$\mathbf{3}_S$	3	1	-1	0	-1
$\boldsymbol{\rho}$	3	-1	-1	0	1
$\boldsymbol{\rho}'$	2	0	2	-1	0

Table 1: Character table of S_4 (see exercise 7.1)

	[(1)]	[(12)(34)]	[(123)]	[(132)]
#elements	1	3	4	4
$\mathbf{1}_T$	1	1	1	1
$\mathbf{1}_1$	1	1	ω	ω^2
$\mathbf{1}_2$	1	1	ω^2	ω
$\boldsymbol{\rho}$	3	-1	0	0

Table 2: Character table of A_4 (see exercise 7.2)