Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 9
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WS 2018/19

Group Theory

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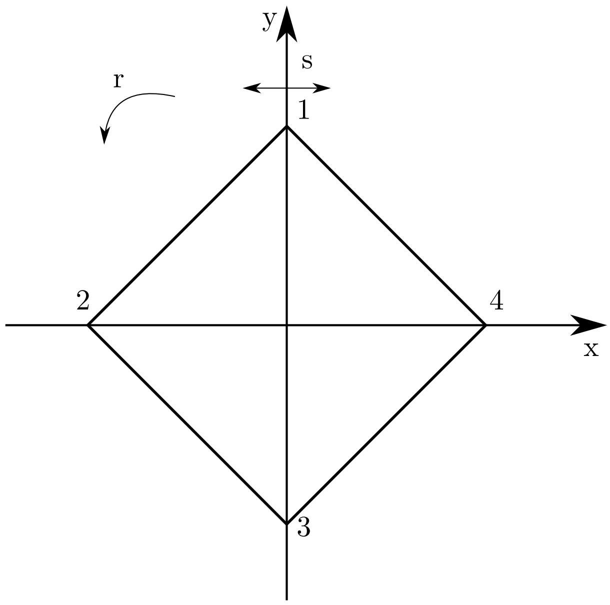
http://www.th.physik.uni-bonn.de/klemm/grouptheory/index.php

Due date: 12/07/2018 — due date on Friday (Dies Academicus)

9.1 Representations of the Dihedral Group D_4

In this exercise we want to look at the irreducible representations of the dihedral group D_4. Recall that the group D_4 is the group of symmetries of a square:



D_4 is generated by reflections s along the y-axis and rotations r by 90°. Algebraically, it can be described by the relations $r^4 = s^2 = e$, $rs = sr^3$, as discussed in the lecture.

i) Determine the number of conjugacy classes of D_4 and the number of elements they contain.

ii) Recall that D_4 is a subgroup of S_4 and determine the embedding: $\phi : D_4 \hookrightarrow S_4$.
<u>*Hint*:</u> Map the generators of D_4 to cycles in S_4 and show that ϕ is a group homomorphism.

iii) Show that the alternating representation $\mathbf{1}_A$ of S_4 restricts to a 1-dimensional representation $\tilde{\mathbf{1}}_A$ of D_4, which is not the trivial representation $\mathbf{1}_T$.

iv) Calculate the number and dimensions of the remaining irreducible representations.

v) Construct a two-dimensional representation $\rho_2 : D_4 \to GL(2, \mathbb{C})$ by realising the generators of D_4 in terms of 2×2 matrices acting geometrically on $\mathbb{R}^2$.

vi) Use the embedding $\phi : D_4 \hookrightarrow S_4$ to calculate the character of the representation $\tilde{\mathbf{3}}_S$ of D_4 obtained from the restriction of the standard representation $\mathbf{3}_S$ of S_4.
<u>*Hint*</u>: The character table of S_4 is given by:

	[(1)]	[(12)]	[(12)(34)]	[(123)]	[(1234)]
#elements	1	6	3	8	6
$\mathbf{1}_T$	1	1	1	1	1
$\mathbf{1}_A$	1	-1	1	1	-1
$\mathbf{3}_S$	3	1	-1	0	-1
$\boldsymbol{\rho}$	3	-1	-1	0	1
$\boldsymbol{\rho}'$	2	0	2	-1	0

vii) Show that $\tilde{\mathbf{3}}_S$ decomposes as $\tilde{\mathbf{3}}_S = \mathbf{2} \oplus \mathbf{1}'$ and compute the character of $\mathbf{1}'$.

viii) Use suitable tensor products to deduce the still remaining representation(s).

ix) Argue that all irreducible representations of D_4 are real.

(12 pts)

9.2 Representations of the Permutation Group S_5 with Young Tableaux

Use the classification of irreducible representations in terms of Young Tableaux. List all irreducible representations of S_5 and compute their dimensions with the hook length formula from the lecture.

(5 pts)

9.3 The Group Algebra $\mathbb{C}[G]$

Let G be a finite group with group algebra $\mathbb{C}[G]$ as defined in the lecture. Let $\mathcal{F}$ be the space of functions, $\mathcal{F} : G \to G$, on which we define a product $* : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ given by:

$$(\alpha * \beta)(g) := \sum_h \alpha(h) \cdot \beta(h^{-1}g) \quad \text{for any } \alpha, \beta \in \mathcal{F}. \tag{1}$$

We identify the space $\mathcal{F}$ with the group algebra $\mathbb{C}[G]$ as:

$$\Phi : \mathcal{F} \longrightarrow \mathbb{C}[G]\ , \quad \alpha \longmapsto \sum_g \alpha(g) e_g\ .$$

Show that the diagram:

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{F} & \xrightarrow{\ *\ } & \mathcal{F} \\ \big\downarrow \Phi & & \big\downarrow \Phi \\ \mathbb{C}[G] \times \mathbb{C}[G] & \xrightarrow{\ \cdot\ } & \mathbb{C}[G] \end{array} \tag{2}$$

commutes, i.e. $\Phi(\alpha * \beta) = \Phi(\alpha) \cdot \Phi(\beta)$.

(3 pts)