Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 1
18.4.2016
SS 16

Exercises General Relativity and Cosmology

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Hand in: 22.4.2016

More precisely, the sheets can be handed in on Friday, the 22.4.2016 until 16:00 in the mailboxes at the door of room 2.011, BCTP. There you will also find the new sheet.

The rooms and tutors for the exercise classes are as follows:

Monday, 10-12	Serminarroom II, HISKP	Urmi Ninad
Wednesday, 10-12	Ü218, Room 2.030, AVZ I	Bardia Farizhendi
Wednesday, 14-16	Seminarroom III, HISKP, Room 1.022	Cesar Fierro Cota

The tutorials start at monday, 25.4.2016.

If you have any questions feel free to contact me under schimann@th.physik.uni-bonn.de. You can find the addresses of the tutors on the course website:

`http://www.th.physik.uni-bonn.de/klemm/grss16/`

–Homework–

1 Minkowski diagrams (10 pts.)

In the following we will work in 1+1 dimensional Minkowski spacetime and set $c = 1$.

a) Draw a spacetime diagram (x, t) and draw
   a) an event.
   b) a light-ray.
   c) the worldline of an object that travels with constant velocity $v < 1$.
   d) the worldline of an object that travels with constant velocity $v > 1$.
   e) the worldline of an accelerated object.

(*2 points*)

b) Draw a spacetime diagram (x, t) of an observer $\mathcal{O}$ at rest. Into this spacetime diagram draw the worldline of an observer $\mathcal{O}'$ that travels with velocity $v < 1$ measured in the rest-frame of $\mathcal{O}$. Draw the coordinate axes of the spacetime diagram of $\mathcal{O}'$.
Hint: What is her time-axis? How do you then construct the space-axis?

(*2 points*)

c) You know that an object with length l' in the frame of the observer $\mathcal{O}'$ appears with length l to the observer $\mathcal{O}$ related to l' by

$$l = \sqrt{1 - v^2} l' . \tag{1}$$

In the following we consider the so-called garage paradox. We consider a car and a garage that have both length l at rest. The garage has a front-door (F) and a back-door (B). It is constructed in such a way, that it opens both doors when the front of the car arrives at the front door, closes both doors, if the back of the car reaches the front-door and opens both doors again, when the car leaves the garage (i.e. the front of the car arrives at the back-door). From the point of view of the garage the car is length-contracted and nicely fits into the garage. From the point of view of the car though, the garage is length-contracted and the car will not fit into it, but one expects that it will be destroyed by the doors. Resolve this paradox.

Hint: Draw a spacetime diagram in which the garage is at rest. What is the order in which the events appear for both observers?

(*4 points*)

d) Draw the spacetime diagram for an observer $\mathcal{O}$ sitting at the origin and draw her light cone. Mark the regions which are in space-like, time-like or light-like distance to him. How does the light cone change when you increase the speed of light? How does it look in the limit when $c \to \infty$?

(*2 points*)

A voluntary group exercise is to grab a bottle of wine and discuss the implications of the absence of a notion of absolute simultaneity for the concept of an objective progression of time. A good starting point is `https://en.wikipedia.org/wiki/B-theory_of_time`. Note also Einsteins words in a letter of condolence to the family of his friend Michele Besso: *"Nun ist er mir auch mit dem Abschied von dieser sonderbaren Welt ein wenig vorausgegangen. Dies bedeutet nichts. Für uns gläubige Physiker hat die Scheidung zwischen Vergangenheit, Gegenwart und Zukunft nur die Bedeutung einer wenn auch hartnäckigen Illusion."*

2 The Lorentz group (15 pts.)

We consider four-dimensional Minkowski space $\mathbb{R}^{1,3}$, which is $\mathbb{R}^4$ equipped with the Minkowski metric η. This is a symmetric, non-degenerate bilinear form $\eta : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ defined by

$$\eta(e_\mu, e_\nu) \equiv \eta_{\mu\nu} = \begin{cases} -1 & \text{for } \mu = \nu = 0 \\ +1 & \text{for } \mu = \nu = 1, 2, 3 \end{cases} \tag{2}$$

for the standard orthonormal basis $\{e_0, e_1, e_2, e_3\}$ on $\mathbb{R}^4$. Using linearity we then find

$$\eta(x, y) = x^{\mathrm{t}} \cdot \tilde{\eta} \cdot y \quad \text{for } x, y \in \mathbb{R}^{1,3}, \tag{3}$$

where $\tilde{\eta}$ is a matrix with entries $\eta_{\mu\nu}$. From now, we identify $\tilde{\eta}$ and η with each other and do not distinguish between them.
For $x, y \in \mathbb{R}^{1,3}$ we write $x \cdot y = \eta(x, y)$ and $x^2 = x \cdot x$. The postulates of special relativity imply that transformations Λ relating two inertial frames, so called Lorentz transformations, preserve the spacetime distance, i.e.

$$(x - y)^2 = (\Lambda(x - y))^2 \quad \text{for all} \quad x, y \in \mathbb{R}^{1,3} . \tag{4}$$

This leads to the definition of the *Lorentz group*

$$\mathrm{O}(1,3) = \{\Lambda \in \mathrm{GL}(4,\mathbb{R}) \,|\, \Lambda^{\mathrm{t}}\eta\Lambda = \eta\}\,. \tag{5}$$

a) Show that $\Lambda \in \mathrm{O}(1,3)$ indeed fulfills eq. (4). *(1 point)*

b) Show that $\mathrm{O}(1,3)$ indeed is a group. *(3 points)*

c) Show that $\Lambda^{\mathrm{t}}\eta\Lambda = \eta$ written in components reads $\eta_{\rho\sigma}\,\Lambda^{\rho}{}_{\mu}\,\Lambda^{\sigma}{}_{\nu} = \eta_{\mu\nu}$. *(1 point)*

d) Embed the group of three-dimensional rotations into $\mathrm{O}(1,3)$. *(1 point)*

e) Show that $|\Lambda^{0}{}_{0}| \geq 1$ and that $|\det\Lambda| = 1$. With this argue that the Lorentz group consists of four branches (which are not continuously connected to each other). Hint: Use $\det(\mathbb{1} + \epsilon\lambda) = 1 + \epsilon\,\mathrm{tr}\,\lambda + \mathcal{O}(\epsilon^2)$. *(3 points)*

f) Show that the subset $\mathrm{SO}^{\uparrow,+}(1,3) = \{\Lambda \in \mathrm{O}(1,3) \,|\, \det\Lambda = 1\,,\, \Lambda^{0}{}_{0} \geq 1\}$ forms a subgroup of $\mathrm{O}(1,3)$, called the *proper orthochronous Lorentz group.*
Hint: To show that $\Lambda''^{0}{}_{0} \geq 1$ for $\Lambda'' = \Lambda\Lambda'$ with $\Lambda, \Lambda' \in \mathrm{SO}^{\uparrow,+}(1,3)$ use the triangle identity for a product of euclidean vectors.
(2 points)

g) Identify the Lorentz transformations for time and parity reversal and relate them to the respective branches. *(1 point)*

Consider two inertial frames, K and K'. When K' moves in K with velocity v in positive x_1 direction, the Lorentz transformation from K to K' is ($c = 1, \gamma = \sqrt{1-v^2}$)

$$\Lambda_{x_1}(v) = \begin{pmatrix} \gamma & -\gamma\cdot v & 0 & 0 \\ -\gamma\cdot v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Transformation of this type are called *boosts.* We introduce the *rapidity* ϕ by $v = \tanh\phi$.

h) Rewrite $\Lambda_{x_1}(v)$ from eq. (6) in terms of the rapidity. *(1 point)*

i) Consider two succsessive boost, both in the x_1 direction but with different velocities. Find the rapidity of the composite boost. Deduce the relativistic rule for addition of velocities. *(2 points)*