Physikalisches Institut
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Theoretische Physik

Exercise Sheet 11
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SS 16

Exercises General Relativity and Cosmology

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`http://www.th.physik.uni-bonn.de/klemm/grss16/`

–Homework–

1 An alternative derivation for the deflection of light (15 pts.)

For the trajectory of a photon you can not use the proper time as a parameter. We can however define a parameter by imposing

$$p^\mu = \frac{dx^\mu}{d\lambda}\,, \tag{1}$$

where p is the 4-momentum of the photon. To derive the deflection of light in the presence of a spherical mass we will again work with the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2MG}{r}\right) dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\Omega^2\,. \tag{2}$$

A sketch of the trajectory with and without deflection is shown in figure 1.

1. Using $p^2 = 0$, derive the relations (we set $G = 1$)

$$\begin{aligned} \frac{dr}{dt} &= \pm\left(1 - \frac{2M}{r}\right)\sqrt{1 - \left(1 - \frac{2M}{r}\right)\frac{(L/E)^2}{r^2}}\,, \\ \frac{d\phi}{dt} &= \pm\frac{L/E}{r^2}\left(1 - \frac{2M}{r}\right)\,, \end{aligned} \tag{3}$$

   where L is the angular momentum and E is the energy of the photon. **3 pts.**

2. Argue that asymptotically the angular momentum of the photon can be written as $L = pb$, where b is the *impact parameter*. What is the interpretation of b? **2 pts.**

3. Use the impact parameter to eliminate the angular momentum and the energy from the equation for the radial velocity. Set the radial velocity equal to zero to determine the radius r_{min}, where the photon is closest to the deflecting mass, in terms of the impact parameter b. **2 pts.**

4. Show that

$$\int_{\phi_i}^{0} d\phi = \frac{1}{r_{min}} \int_0^1 \frac{du}{\frac{1}{b^2} - \frac{u^2}{r_{min}^2}\left(1 - \frac{2M}{r_{min}}u\right)}\,, \tag{4}$$

   by replacing the integral over $d\phi$ by the corresponding integral over dr and substitute $u = r_{min}/r$. **3 pts.**

5. For stars we can normally assume $r_{min} \gg 2M$. Introduce $x = M/r_{min} \ll 1$ and Taylor expand the integrand to first order in x. **3 pts.**

6. Solve the resulting integrals using a computer algebra system to obtain the total deflection

$$\delta\phi = \frac{4M}{r_{min}}\,. \tag{5}$$

   2 pts.

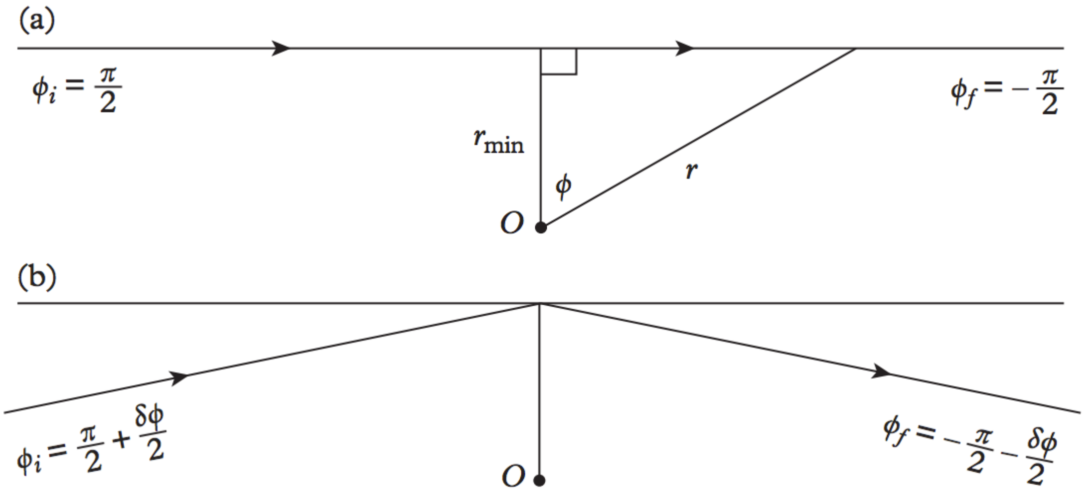

Figure 1: Trajectory of the deflected and undeflected light ray.