Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 2
22.4.2016
SS 16

Exercises General Relativity and Cosmology

Prof. Dr. Albrecht Klemm, Thorsten Schimannek

Hand in: 29.4.2016

Just a reminder: The solutions must be handed in in groups of two to three students.

http://www.th.physik.uni-bonn.de/klemm/grss16/

–Homework–

1 Tangents to the sphere (15 pts.)

In the lecture you learned about the stereographic projection as a method to equip the sphere via an embedding into $\mathbb{R}^3$ with an atlas consisting of two charts. The goal of this exercise is to use this example to do some hands on calculations with charts, coordinates and tangent vectors.

1. Although you should think about a manifold as independent of an embedding we will use the embedding of the sphere into $\mathbb{R}^3$ to construct coordinate charts and visualize the otherwise intrinsic concepts. Consider the unit sphere $S^2 \subset \mathbb{R}^3$ defined by $x^2+y^2+z^2=1$. Use the north pole $p_n=(0,0,1)$ and assign a point on $p \in S^2 \backslash n_p$ to the point in the plane $z=-1$ which is the intersection of $z=-1$ with the ray connecting p and p_n. Show that in coordinates the map is given by
$$\begin{aligned} \phi_1 &: U_1 \to \mathbb{R}^2 \\ (x,y,z) &\mapsto \left(\frac{2x}{1-z}, \frac{2y}{1-z}\right), \end{aligned} \tag{1}$$
with $U_1 = S^2 \backslash p_n$. Do an analogue construction for the south pole and the plane $z=1$ and give the corresponding map $\phi_2 : S^2 \backslash (0,0,-1) \to \mathbb{R}^2$.
(2 points)

2. Construct the inverses of ϕ_1, ϕ_2.
(3 points)

3. What is the condition for two charts to be compatible? Explicitly show that the charts (U_1,ϕ_1) and (U_2,ϕ_2) are compatible on the intersection.
(3 points)

4. Consider the tangent space to the point $\phi_1^{-1}(p)$ in the chart U_1. You know from the lecture that the coordinate basis of tangent vectors is given by ∂_x and ∂_y. How do these tangent vectors embed into the tangent space of $\mathbb{R}^3$ at $\phi_1^{-1}(p)$ for $p=(0,0)$, $(1,0)$ and $(2,0)$? Sketch the $y'=0$ plane of the sphere and draw the stereographic projection of these points and the corresponding tangent vectors. What is the norm of these vectors using the standard euclidean metric on $\mathbb{R}^3$?
(3 points)

5. How do you need to restrict the domain of θ and ϕ so that the spherical coordinates

$$\phi_s^{-1}(\phi, \theta) : \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} , \tag{2}$$

provide a chart on the sphere?

(1 point)

6. Choose a domain for θ and ϕ such that $\phi_s^{-1}(0, \pi/2)$ is in the corresponding chart. Transform the vector ∂_θ at $(\phi, \theta) = (0, \pi/2)$ into the coordinate basis corresponding to the chart (U_1, ϕ_1) given above.

(3 points)

2 Differential forms and electromagnetism (10 pts.)

From the lecture you know that a differential p-form is a $(0, p)$-tensor field which is completely antisymmetric. The space of all p-forms on a manifold M is called $\Lambda^p(M)$. The exterior derivative

$$d : \Lambda^p(M) \to \Lambda^{p+1}(M) \tag{3}$$

acts on a p-form $A = A_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$ as

$$dA = (p+1)\partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1}} . \tag{4}$$

Note that the action of the exterior derivative is independent of the metric.

1. Show that the exterior derivative of a p-form transforms as a $(0, p+1)$ tensor.

(3 points)

On an n-dimensional manifold M the Hodge star operator

$$\star : \Lambda^p(M) \to \Lambda^{n-p}(M) , \tag{5}$$

acts like

$$\star A = \frac{1}{p!} \epsilon^{\nu_1 \ldots \nu_p}{}_{\mu_1 \ldots \mu_{n-p}} A_{\nu_1 \ldots \nu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{n-p}} . \tag{6}$$

It depends on the metric due to the definition of the Levi-Civita tensor and the need to raise and lower indices.

2. First consider 3-dimensional, flat euclidean space and write $v = v_x dx + v_y dy + v_z dz$. Express the divergence

$$\vec{\nabla} \cdot \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \partial_x v_x + \partial_y v_y + \partial_z v_z , \tag{7}$$

and the rotation

$$\vec{\nabla} \times \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \partial_y v_z - \partial_z v_y \\ \partial_z v_x - \partial_x v_z \\ \partial_x v_y - \partial_y v_x \end{pmatrix} , \tag{8}$$

as an action on v in terms of the exterior derivative and the Hodge star operator.

(3 points)

3. We now move to 4-dimensional Minkowski space. The electromagnetic tensor F can be written as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{9}$$

Show that the Maxwell equations can be written as

$$dF = 0, \quad d \star F = \star J \,, \tag{10}$$

with $J = J_t dt + J_x dx + J_y dy + J_z dz$ being the electromagnetic 4-current, written as a $(0, 1)$-tensor.

(4 points)